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Orthogonal polynomials and Painleve equations : Deformed Laguerre weight function and Painleve equation V (PV).

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Orthogonal polynomials and Painlevé equations: Deformed Laguerre weight function and Painlevé equation V (P_V).

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Dedication

To

my Parents:

Vénant NDAYIRUKIYE

and

Monique NIZIGAMA,

my Brothers and my Sister,

all my Familly

and

all my Acquaintances.

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Abstract

In this work, we focus on the link between orthogonal polynomials and Painlevé equations. We investigate the orthogonal polynomials with respect to the deformed Laguerre weight function $w(x,t) = x^\alpha (x+1)^\beta e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0$. By applying the Ladder operators approach to our weight function, we show that the recurrence coefficients of monic polynomial with respect to this weight are in terms of auxiliary quantities $R_n(t)$ and $r_n(t)$ that satisfy the coupled Ricatti equations from which we find that $R_n(t)$, up to a certain linear fractional transformation, satisfies a particular fifth Painlevé equation $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$. This is the main contribution of our work and is contained in the Theorem 3.6.

Keywords: Orthogonal polynomials, Painlevé equations, Laguerre polynomials, Ladder operators.

Résumé

Dans ce travail, nous nous focalisons sur le lien entre les polynômes orthogonaux et les équations de Painlevé. Nous étudions les polynômes orthogonaux par rapport à la fonction poids de Laguerre déformée $w(x,t) = x^{\alpha}(x+1)^{\beta}e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0$. En appliquant l'approche des opérateurs d'échelle à notre fonction de poids, nous montrons que les coefficients de récurrence pour les polynômes moniques par rapport à cette fonction de poids sont en fonction des quantités auxiliaires $R_n(t)$ et $r_n(t)$ qui satisfont un couple des équations de Ricatti à partir desquelles nous trouvons que $R_n(t)$, à une certaine transformation fractionnaire linéaire, satisfait une cinquième équation particulière de Painlevé $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$. Ceci constitue la contribution principale de notre travail et est contenue dans le Théorème 3.6.

Mots clés: Polynômes orthogonaux, équations de Painlevé, Polynômes de Laguerre, opérateurs d'échelle.

Condensé en Français

Polynômes orthogonaux sur l'axe réel

Les polynômes orthogonaux $p_n(x)$ sur l'intervalle (a, b) satisfont la relation suivante

$$< p_m, p_n >= \int_a^b p_m(x)p_n(x)w(x)dx = h_n\delta_{m,n}, \quad m, n > 0.$$

Ces polynômes satisfont aussi la relation de récurrence à trois termes

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + c_np_{n-1}(x).$$

Pour les polynômes orthogonaux moniques $P_n(x) = p_n/k_n$ (où $p_n(x) = k_nx^n + ...$) la relation précédente devient

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \geq 1.$$

Les polynômes orthogonaux peuvent avoir une représentation matricielle. Les polynômes moniques sont représentés par

$$P_n(x) = \frac{1}{D_n} det \begin{pmatrix} m_0 & m_1 & m_2 & ... & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & ... & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & ... & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & ... & m_{2n-2} & m_{2n-1} \\ 1 & x & x^2 & ... & x^{n-1} & x^n \end{pmatrix},$$

où D_n est le déterminant de Hankel défini par

$$D_n = det \begin{pmatrix} m_0 & m_1 & m_2 & ... & m_{n-2} & m_{n-1} \\ m_1 & m_2 & m_3 & ... & m_{n-1} & m_n \\ m_2 & m_3 & m_4 & ... & m_n & m_{n+1} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & ... & m_{2n-3} & m_{2n-2} \\ \end{pmatrix}$$

pourvu que les moments $m_k := \int_a^b x^k w(x)dx, \quad k = 0, 1, 2, ...$ existent.

Nous résumons les polynômes orthogonaux classiques dans le tableau suivant

Name	$p_n(x)$	$w(x)$	(a, b)	Condition sur les paramètres
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty, \infty)$	$-$
Laguerre	$L_n^{(\alpha)}(x)$	$x^\alpha e^{-x}$	$(0, \infty)$	$\alpha > -1$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha(1+x)^\beta$	$(-1, 1)$	$\alpha, \beta > -1$
Legendre	$P_n(x)$	1	$(-1, 1)$	$-$
Tchebychev	$T_n(x)$ or $U_n(x)$	$(1-x^2)^{-\frac{1}{2}}$	$(-1, 1)$	$-$
Gegenbauer	$P_n^{(\lambda)}(x)$	$(1-x^2)^{\lambda-\frac{1}{2}}$	$(-1, 1)$	$\lambda > -\frac{1}{2}$ and $\lambda - \frac{1}{2} \neq \frac{1}{2}$

Les polynômes orthogonaux classiques et semi-classiques satisfont une propriété connue sous le nom de relation de structure

$$\sigma(x)p_n'(x) = \sum_{k=n-t}^{n+s-1} A_{n,k}p_k(x),$$

où $\sigma(x)$ satisfait l'équation de Pearson

$$[\sigma(x)w(x)]' = \tau(x)w(x)$$

et $s = deg\sigma(x), \quad t = max\{deg\tau(x),\ deg\sigma(x) - 1\}$.

La relation de récurrence à trois termes et la relation de structure sont toujours compatibles.

Quelques propriétés des polynômes de Laguerre

Les polynômes de Laguerre peuvent être définis au moyen de la formule de Rodrigues par

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right), \quad n \in \mathbb{N}.$$

D'une façon explicite, ils sont donnés par

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \quad n \in \mathbb{N}.$$

La condition d'orthogonalité pour les polynômes de Laguerre est donnée par

$$\int_0^\infty L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)x^\alpha e^{-x}dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{m,n}, \quad m,n \geq 0.$$

La relation de recurrence à trois termes pour les polynômes moniques de Laguerre peut être obtenue en utilisant la compatibilité entre la formule de récurrence à trois termes et la relation de structure. Elle est donnée par

$$xP_n(x) = P_{n+1}(x) + (2n+\alpha+1)P_n(x) + n(n+\alpha)P_{n-1}(x),$$

où $P_n(x) = L_n^{(\alpha)}/k_n$ avec $k_n = (-1)^n/n!$.

Equations de Painlevé

Découvertes par Painlevé [47, 48]et Gambier [21] entre la fin du 19$^{\text{è}}$ et le début du 20$^{\text{è}}$ siècles, les équations de Painlevé sont des équations différentielles non linéaires du second ordre de la forme

$$y'' = R(y', y, x), \quad R \text{ rationnelle},$$

qui possèdent la propriété de Painlevé.
Propriété de Painlevé : La solution générale de l'équation précédente ne dépend pas d'une branche mobile de points.

Painlevé et Gambier ont trouvé 50 types d'équations qui possèdent la propriété de Painlevé dont 44 qui peuvent être réduites aux équations déjà connues et les six nouvelles équations connues sous le nom des équations de Painlevé.

Ces six équations de Painlevé sont :

$$P_I \quad y'' = 6y^2 + x,$$

$$P_{II} \quad y'' = 2y^3 + xy + \alpha,$$

$$P_{III} \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y},$$

$$P_{IV} \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y},$$

$$P_V \quad y'' = (\frac{1}{2y} + \frac{1}{y-1})(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1},$$

$$P_{VI} \quad y'' = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x})(y')^2 - (\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x})y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right),$$

où α, β, γ, δ sont des constantes.
Les solutions des équations de Painlevé sont appelées des *transcendantes de Painlevé*.
Dans notre travail, nous allons nous intéressé sur la cinquième équation P_V.

Equations discrètes de Painlevé

Les équations discrètes de Painlevé sont des équations de récurrence non linéaires du second ordre dont la limite continue donne l'une des six équations différentielles de Painlevé.
La liste partielle de ces équations est :

$$d\text{-}P_I \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + a(-1)^n}{x_n} + b,$$

$$d\text{-}P_{II} \quad x_{n+1} + x_{n-1} = \frac{x_n z_n + a}{1 - x_n^2},$$

$$d\text{-}P_{IV} \quad (x_{n+1}+x_n)(x_n+x_{n-1}) = \frac{(x^2-a^2)(x_n^2-b^2)}{(x_n+z_n)^2-c^2},$$

$$d\text{-}P_V \quad \frac{(x_{n+1}+x_n-z_{n+1})(x_n+x_{n-1}-z_n-z_{n-1})}{(x_{n+1}+x_n)}(x_n+x_{n-1})$$

$$= \frac{[(x_n-z_n)^2-a^2][(x_n-z_n)^2-b^2]}{(x_n-c^2)(x_n-d^2)},$$

où $z_n = \alpha n + \beta$ et a, b, c, d des constantes.
Dans ces équations de la liste précédente, x_n et x_{n+1} apparaissent sous forme additive. Il existe d'autres équations discrètes connues sous le nom d'équations q-discrètes de Painlevé où x_n et x_{n+1} apparaissent sous forme multipicative.
Comme exemples, nous avons:

$$q\text{-}P_{III} \quad x_{n+1}x_{n-1} = \frac{(x_n-aq_n)(x_n-bq_n)}{(1-cx_n)(1-x_n/c)},$$

$$q\text{-}P_V \; (x_{n+1}x_n-1)(x_{n+1}x_{n-1}-1) = \frac{(x_n-a)(x_n-1/a)(x_n-b)(x_n-1/b)}{(1-cx_nq_n)(1-x_nq_n/c)},$$

$$q\text{-}P_{VI} \quad \frac{(x_nx_{n+1}-q_nq_{n+1})(x_nx_{n-1}-q_nq_{n-1})}{(x_nx_{n+1}-1)(x_nx_{n-1}-1)}$$

$$= \frac{(x_n-aq_n)(x_n-q_n/a)(x_n-bq_n)(x_n-q_n/b)}{(x_n-c)(x_n-1/c)(x_n-d)(x_n-1/d)},$$

où $q_n = q_0 q^n$ et a, b, c, d sont des constantes.

Il existe aussi d'autres formes d'équations discrètes de Painlevé. Pour plus d'exemples, voir notre annexe.

Fonction de poids de Laguerre deformée et P_V

Nous considérons la fonction de poids de Laguerre déformée

$$w(x,t) = x^\alpha (x+1)^\beta e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0.$$

En appliquant la technique des opérateurs d'échelle, nous obtenons deux quantités auxiliaires $R_n(t)$ et $r_n(t)$ en lesquelles sont exprimé les coefficients de récurrence pour les polynômes moniques orthogonaux par rapport à cette fonction de poids en haut et nous montrons que $R_n(t)$ et $r_n(t)$ satisfont un couple de deux équations de Ricatti, à partir desquelles nous trouvons que $R_n(t)$ satisfait l'équation de Painlevé $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$.
Nous signalons que Basor et Chen [7] ont étudié une autre fonction de poids de Laguerre déformée $w(x,t) = x^\alpha (x+t)^\lambda e^{-x},\ \alpha > -1,\ t > 0,\ x > 0$ qui est proche de la nôtre. Ils ont aussi trouvé une équation de Painlevé V mais en utilisant une transformation de $R_n(t)$ qui est différente de la nôtre.

Pour notre fonction de poids, la condition d'orthogonalité devient

$$\int_0^\infty P_m(x,t)P_n(x,t)x^\alpha (x+1)^\beta e^{-tx}dx = h_n(t)\delta_{m,n},\ h_n(t) > 0,$$

et la relation de récurrence à trois termes devient

$$xP_n(x,t) = P_{n+1}(x,t) + b_n(t)P_n(x,t) + c_n(t)P_{n-1}(x,t),$$

où

$$P_n(x,t) = x^n + p(n,t)x^{n-1} + \dots\ ,$$

avec $P_0(x,t) = 1,\ c_0(t)P_{-1}(x,t) = 0$.
Nous avons aussi le déterminant de Hankel qui est donné par

$$\begin{aligned} D_n : &= det\left(\int_0^\infty x^{i+j+\alpha}(x+1)^\beta e^{-tx}dx\right)_{i,j=0}^{n-1} \\ &= \prod_{j=0}^{n-1} h_j(t). \end{aligned}$$

Opérateurs d'échelle et ses conditions supplémentaires

L'approche par opérateurs d'échelle est l'une des techniques la plus utilisée pour trouver les coefficients de récurrence pour les polynômes orthogonaux et dans l'étude des déterminant de Hankel qui jouent un rôle fondamental dans la théorie des matrices aléatoires.
Nous résumons ces opérateurs d'échelle et ses conditions supplémentaires par les trois théorèmes suivants:

Theorem 0.1. *Soit $w(x)$ une fonction de poids lisse définie sur (a, b) telle que $w(a) = w(b) = 0$. Les polynômes orthogonaux moniques par rapport à $w(x)$ satisfont l'équation d'opérateur d'abaissement*

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = c_n A_n(z) P_{n-1}(z),$$

où

$$A_n(z) := \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(x)}{z - x} P_n^2(x) w(x) dx, \tag{1}$$

$$B_n(z) := \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(x)}{z - x} P_n(x) P_{n-1}(x) w(x) dx \tag{2}$$

et $v(z) := -\ln w(z)$.

Theorem 0.2. *Les fonctions $A_n(z)$ et $B_n(z)$ définies comme dans (1) et (2) satisfont les équations suivantes:*

$$B_{n+1}(z) + B_n(z) = (z - b_n) A_n(z) - v'(z), \tag{S_1}$$

$$c_{n+1} A_{n+1} - c_n A_{n-1} = 1 + (z - b_n)[B_{n+1}(z) - B_n(z)], \tag{S_2}$$

$$B_n^2(z) + v'(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = c_n A_n(z) A_{n-1}(z) \qquad (S_2'),$$

Theorem 0.3. *Les polynômes orthogonaux moniques $P_n(z)$ satisfont l'équation de l'opérateur de relèvement*

$$\left(\frac{d}{dz} - B_n(z) - v'(z)\right) P_{n-1}(z) = -A_n(z) P_n(z),$$

où $A_n(z)$ et $B_n(z)$ sont définies comme dans (1) et (2).

Nous appliquons ces théorèmes à notre fonction de poids et nous produisons d'abord les deux propositions suivantes:

Proposition 0.1 *Pour notre problème, nous avons*

$$A_n(z) = \frac{R_n}{z} + \frac{t - R_n}{z+1},$$

$$B_n(z) = \frac{r_n}{z} - \frac{n + r_n}{z+1},$$

où

$$R_n := \frac{\alpha}{h_n}\int_0^\infty \frac{P_n^2(x)w(x)}{x}dx,$$

$$r_n := \frac{\alpha}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x}dx.$$

Proposition 0.2 *Les quantités R_n et r_n satisfont les équations discrètes de Painlevé suivantes:*

$$t(r_{n+1} + r_n) = -R_n^2 - (2n + 1 + \alpha + \beta - t)R_n + \alpha t,$$

$$n(n+\beta) + (2n+\alpha+\beta)r_n = r_n(r_n - \alpha)\left(\frac{t}{R_n} - \frac{t^2}{R_nR_{n-1}} + \frac{t}{R_{n-1}}\right).$$

Puis, en tenant compte de la dépendance du paramètre t, nous obtenons le lemme suivant:

Lemma 0.4. *Les coefficients de récurrence b_n et c_n satisfont les équations différentielles suivantes*

$$\frac{dc_n}{dt} = c_n(b_{n-1} - b_n),$$

$$\frac{db_n}{dt} = c_n - c_{n+1}.$$

Ensuite, avec l'aide du lemme précédent et le déterminant de Hankel défini en haut, nous obtenons le lemme suivant:

Lemma 0.5. *Les quantités auxiliaires R_n et r_n satisfont les équations de Riccati couplées suivantes :*

$$t\frac{dR_n}{dt} = -\alpha t + (2n + 1 + \alpha + \beta - t)R_n + R_n^2 + 2tr_n,$$

$$\frac{dr_n}{dt} = \frac{r_n^2 - \alpha r_n}{R_n} - \frac{R_n}{t(t-R_n)}\left[n(n+\beta) + (2n+\alpha+\beta)r_n + \frac{t}{R_n}(r_n^2 - \alpha r_n)\right],$$

où R_n et r_n sont définies en (1) et (2), respectivement.

Enfin, grâce à ce lemme nous arrivons au résultat principal de notre travail que nous présentons au moyen du théorème suivant:

Theorem 0.6. *La quantité R_n satisfait l'équation différentielle non linéaire d'ordre 2 suivante*

$$R_n'' = \left(\frac{1}{R_n} - \frac{1}{t-R_n}\right)\frac{(R_n')^2}{2} + \left(-1 + \frac{t}{t-R_n}\right)\frac{R_n'}{t} + (R_n + 2n + 1 + \alpha + \beta)\frac{R_n^2}{t^2}$$

$$+ \left[\frac{-3R_n}{2} - (2n+1+\alpha+\beta)\right]\frac{R_n}{t} + \frac{R_n}{2} + \frac{\beta^2 - 1}{2(t-R_n)} - \frac{\alpha^2}{2R_n} + \frac{\alpha^2 - \beta^2 + 1}{2t}.$$

Soit $y(t) := 1 - \frac{t}{R_n(t)}$*, alors* $y(t)$ *satisfait une équation différentielle du second ordre*

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2}{2}y - \frac{\beta^2/2}{y}\right) + (2n+1+\alpha+\beta)\frac{y}{t} - \frac{y(y+1)}{2(y-1)},$$

qui est une équation particulière de Painlevé V, c'est-à-dire, $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$.

Conclusion

Nous rappelons que notre travail s'intéresse sur le lien entre les polynômes orthogonaux et les équations de Painlevé. Plus précisément, notre objectif principal était de trouver une équation de Painlevé à partir d'une fonction de poids donnée. Nous revenons sur quelques propriétés des polynômes orthogonaux sur la l'axe réel et aussi sur les équations de Painlevé. Nous atteignons notre objectif principal en étudiant les polynômes orthogonaux par rapport à la fonction de Laguerre deformée et nous montrons dans le Théorème 0.6 que cette fonction de poids satisfait la cinquième équation de Painlevé qui est le résultat principal de ce travail.

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Preface

This thesis is realized within the framework of the obtaining of the Diploma of Master in Fundamental and Applied Mathematics.

The idea for this research was born when I met Professor Walter Van Assche at the University of Burundi in 2019 while he was teaching me the course of Theoretical Physics and Mathematics in Master I. I was inspired by the way he introduced me to orthogonal polynomials and their applications. I then became interested in his research works, especially in orthogonal polynomials and Painlevé equations.

This work is a continuation of the works of researchers interested in relationship between orthogonal polynomials and Painlevé equations. In this work, the problem of finding a Painlevé equation from a deformed Laguerre weight function is solved by using the Ladder operators approach.

Introduction

The investigation of relationship between orthogonal polynomials and Painlevé equations is very recent. On one hand, we find many topics dealing with orthogonal polynomials showing that the latter have been widely investigated by many mathematicians since the late of 19th century [12, 27, 51]. Orthogonal polynomials appear very useful in practice in various domains of mathematics (numerical analysis, probability theory, theory of random matrices, representation theory of Lie groups, number theory, etc), engineering, physics. Furthermore, with the advent of computers, they have become tools of approximation and encoding-decoding [56, 57]. That is why their investigation stays interesting to deepen until today. On the other hand, we find some topics dealing with Painlevé equations. However, there are few works showing the link between orthogonal polynomials and Painlevé equations. In [54], one meets some cases which show how some systems of orthogonal polynomials can be described by Painlevé equations. In addition, we find some papers showing how Painlevé equation can be found from a given weight function [6–8, 58] and until today many researchers are working on this topic.

The main contribution of this work is to find a system of orthogonal polynomials that can be described by a Painlevé equation, more precisely, to find a Painlevé equation from a given weight function for orthogonal polynomials. This result is contained in the Theorem 3.6.

In Chapter 1 , we talk about orthogonal polynomials on the real line. As more information on this topic is widely available in the literature, we will only return to some facts that will be very necessary for understanding of our work.

Chapter 2 concerns the Painlevé equations. We start by giving a little history of Painlevé equations, their definition and some of their mathematical properties especially for the fifth Painlevé equation P_V.

Chapter 3 is devoted to the main results of our work. We will first describe the Ladder operators approach that we will use to solve our problem. Next, we get the heart of the matter and we show that from a given deformed Laguerre weight function, the system of orthogonal polynomials with respect to this weight function can be described by a particular Painlevé equation which is P_V.
Finally, we end with the conclusion.

Chapter 1

Orthogonal polynomials on the real line

In this chapter, we will return to some facts about orthogonal polynomials on the real line but not in exhaustive way because here, we will mainly limit ourselves to what we will need to use in this work.
Although we wanted to work a lot more on orthogonal polynomials on the real line in this chapter, let us note that orthogonal polynomials can also be defined on the unit circle [51, Chap.XI],[50] or on the arbitrary curve[51, Chap.XVI].

1.1 Definition of orthogonal polynomials

Let $a, b \in \mathbb{R}$ and $f(x), g(x)$ be two functions in the Hilbert space $L^2[a, b]$.

We recall that the Hilbert space is the inner product space which is complete.
Consider the inner product in $L^2[a, b]$ defined by

$$< f, g >:= \int_a^b f(x)g(x)w(x)dx \ ,$$

where $w(x)$ is a positive function, called weight function.
This is a special case of the inner product

$$< f, g >:= \int_{\mathbb{R}} f(x)g(x)d\mu(x),$$

where μ is a positive measure on $\mathbb{R}$, namely the case $d\mu(x) = w(x)dx$ on an interval (a, b).

Definition 1.1. A sequence of polynomials $\{p_n(x)\}_n^\infty$ with degree $[p_n(x)] = n$ for each n is called orthogonal with respect to the weight function $w(x)$ on the interval (a, b) with $a < b$ if

$$< p_m, p_n >= \int_a^b p_m(x)p_n(x)w(x)dx = h_n\delta_{m,n}, \tag{1.1}$$

where

$h_n \neq 0$ is normalization constant and

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

The interval (a, b) is called the interval of orthogonality and it may be finite or not. Furthermore, the weight function $w(x)$ must be positive on the interval (a, b) such that moments

$$m_k := \int_a^b x^k w(x)dx, \quad k = 0, 1, 2, ... \quad . \tag{1.2}$$

exist [12, 51].

Definition 1.2.

- If h_n in (1.1) equals to one for each $n \in \{0, 1, 2, ...\}$, the sequence of polynomials is called orthonormal.
- If the leading coefficient of p_n is equal to one, i.e $p_n(x) = x^n+$ lower order terms, for each $n \in \{0, 1, 2, ...\}$, the polynomials are called monic.

In what follows, to distinguish monic from orthogonal polynomials $p_n(x)$ in general case, we will denote monic orhogonal polynomials by $P_n(x)$.

1.2 Three-term recurrence relation

It is important to know that any sequence of orthogonal polynomials can be obtained using a three-term recurrence relation, especially when one wants to do numerical computation.

Theorem 1.3. *[31] A sequence of orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ always satisfies*

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + c_np_{n-1}(x), \tag{1.3}$$

with $c_0p_{-1} = 0$ and $a_n,\ b_n,\ c_n$ real constants,
where $a_{n+1} = \frac{k_n}{k_{n+1}},\ c_n = a_n\frac{h_n}{h_{n-1}},\ a_nc_n > 0$ and k_n is the leading coefficient of $p_n(x)$.

Proof. Let $a_{n+1} = \frac{k_n}{k_{n+1}}$.
This implies that the terms in x^{n+1} of the polynomial $a_{n+1}p_{n+1}(x) - xp_n(x)$ vanish so that $a_{n+1}p_{n+1}(x) - xp_n(x)$ is a polynomial of degree $\leq n$.
Hence,

$$a_{n+1}p_{n+1}(x) - xp_n(x) = \sum_{i=0}^{n} \alpha_ip_i(x).$$

The orthogonality property yields

$$< a_{n+1}p_{n+1}(x) - xp_n(x), p_k >= \sum_{i=0}^{n} \alpha_i < p_i(x), p_k(x) >= \alpha_kh_k,$$

that is

$$\begin{aligned}
\alpha_kh_k &= < a_{n+1}p_{n+1}(x) - xp_n(x), p_k > \\
&= - < xp_n(x), p_k(x) > \\
&= - < p_n(x), xp_k(x) > .
\end{aligned}$$

For $k < n-1$, the polynomial $xp_k(x)$ is of degree $< n$ and this implies

$$< p_n(x), xp_k(x) >= 0.$$

Hence,
$\alpha_k = 0$ for $k < n-1$.
Therefore,

$$a_{n+1}p_{n+1}(x) - xp_n(x) = \alpha_np_n(x) + \alpha_{n_{n-1}}p_{n-1}(x).$$

Moreover, from $\alpha_k h_k = - < p_n(x), p_k(x) >$, for $k = n-1$, we have

$$\begin{aligned} \alpha_{n-1} h_{n-1} &= - < p_n(x), x p_{n-1}(x) > \\ &= -a_n h_n, \end{aligned}$$

which implies

$$\alpha_{n-1} = -a_n \frac{h_n}{h_{n-1}} := -c_n.$$

which again implies

$$a_n c_n = a_n^2 h_n / h_{n-1} > 0;$$

and similarly for $k = n$, we have

$$\alpha_n = -\frac{< p_n(x), x p_n(x) >}{h_n} := -b_n.$$

This ends the proof. □

Remark 1.4.

1. For orthonormal polynomials, the three-term recurrence relation (1.3) takes the form

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geq 1 \quad , \tag{1.4}$$

   with $c_0 p_{-1} = 0$.

2. For monic orthogonal polynomials, the three-term recurrence relation in (1.3) becomes

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1 \quad , \tag{1.5}$$

   with $c_0 P_{-1} = 0$

   Furthermore, the Fourier coefficients b_n and c_n in the equation (1.5) can be expressed as

$$b_n = \frac{1}{h_n} \int_a^b x P_n^2(x) w(x) dx \tag{1.6}$$

$$c_n = \frac{1}{h_{n-1}} \int_a^b x P_n(x) P_{n-1}(x) w(x) dx \tag{1.7}$$

or

$$c_n = h_n/h_{n-1}. \tag{1.8}$$

3. If orthonormal polynomials $p_n(x)$ satisfy (1.4), then the corresponding monic polynomials $P_n = p_n/k_n$ satisy (1.5) with $c_n = a_n^2$. That is

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x). \tag{1.9}$$

4. If the weight function $w(x)$ is even ($w(-x) = w(x)$) on a symmetric interval $(-a, a)$, then $b_n = 0$.

The converse of the previous theorem is also true and it is known as spectral theorem of orthogonal polynomials [54] or Favard theorem [12, Th.4.4]. Hence, this rise to two important problems:

Problem 1: It is known as direct problem for orthogonal polynomials and consists, from a known measure μ, in knowing what can be said about the recurrence coefficients $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 1}$.

Problem 2: It is known as the inverse problem for orthogonal polynomials and consists, from recurrence coefficients $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 1}$ assumed to be known, in knowing what can be said about the orthogonality measure μ.

It is also important to end this section by pointing out that the three-term recurrence relation for monic orthogonal polynomials will be fundamental to arrive at the main result of this thesis.

1.3 Hankel determinant and matrix representation of orthogonal polynomials

The monic orthogonal polynomials can be expressed in terms of moments and this can leads to the partial solution of problem 1 cited in the previous section.
When the moments in (1.2) are all positive, the Hankel determinant defined by

$$D_n = det\left((m_{i+j})\right)_{i,j=0}^{n-1}, \tag{1.10}$$

which is explicitly

$$D_n = det\begin{pmatrix} m_0 & m_1 & m_2 & \dots & m_{n-2} & m_{n-1} \\ m_1 & m_2 & m_3 & \dots & m_{n-1} & m_n \\ m_2 & m_3 & m_4 & \dots & m_n & m_{n+1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-3} & m_{2n-2} \end{pmatrix},$$

is positive, and then the monic orthogonal polynomial $P_n(x)$ is given by [17, 55],

$$P_n(x) = \frac{1}{D_n} det\begin{pmatrix} m_0 & m_1 & m_2 & \dots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \dots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \dots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-2} & m_{2n-1} \\ 1 & x & x^2 & \dots & x^{n-1} & x^n \end{pmatrix}. \tag{1.11}$$

Furthermore, the recurrence coefficients in (1.4) are expressed as follows

$$b_n = \frac{D^*_{n+1}}{D_{n+1}} - \frac{D^*_n}{D_n}, \tag{1.12}$$

where D^*_n is obtained from D_n by replacing the last column $(m_{n-1}, m_n, ..., m_{2n-2)^T}$ by $(m_n, m_{n+1}, ..., m_{2n-1})^T$ and

$$c_n = \frac{D_{n-1}D_{n+1}}{D_n^2} > 0. \tag{1.13}$$

Even if the formulas (1.12) and (1.13) show that recurrence coefficients can be computed in terms of Hankel determinant, they do not really show how properties of the measure μ can be transferred to properties of the recurrence coefficients. One needs more tools to solve this direct problem for orthogonal polynomials. For the inverse problem for orthogonal polynomials, it can be solved by passing by a so-called Jacobi matrix or Jacobi operator and if this matrix is self-adjoint, its spectral measure is precisely the orthogonality measure μ [54]. In this work, we will solve this direct problem for orthogonal polynomials for a special case from a particular given weight function. We will see that the Hankel determinant defined in (1.10) will also play a very important role in this thesis.

1.4 Classical orthogonal polynomials

The classical orthogonal polynomials are named after the ones who discovered them [12, 55]. Those are Hermite, Laguerre and Jacobi (including the special cases named after Tchebychev, Legendre and Gegenbauer (or Ultraspherical)) polynomials. We collect them in the following table:

Name	$p_n(x)$	$w(x)$	(a,b)	Condition on parameters
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty,\infty)$	–
Laguerre	$L_n^{(\alpha)}(x)$	$x^\alpha e^{-x}$	$(0,\infty)$	$\alpha > -1$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha(1+x)^\beta$	$(-1,1)$	$\alpha,\beta > -1$
Legendre	$P_n(x)$	1	$(-1,1)$	–
Tchebychev	$T_n(x)$ or $U_n(x)$	$(1-x^2)^{-\frac{1}{2}}$	$(-1,1)$	–
Gegenbauer	$P_n^{(\lambda)}(x)$	$(1-x^2)^{\lambda-\frac{1}{2}}$	$(-1,1)$	$\lambda > -\frac{1}{2}$ and $\lambda - \frac{1}{2} \neq \frac{1}{2}$

TABLE 1.1: *Classical orthogonal polynomials*

All the classical orthogonal polynomials satisfy the orthogonality condition in (1.1), the three-term recurrence relation in (1.3), a second differential order linear differential equation, a so-called Rodrigues formula and a structure relation to which we will return in the next section.

1.5 Structure relation of classical and semi-classical orthogonal polynomials

The weights of classical orthogonal polynomials on the real line satisfy a first order differential equation known as the Pearson equation:

$$[\sigma(x)w(x)]' = \tau(x)w(x), \tag{1.14}$$

where $\sigma(x)$ is a polynomial of degree at most 2 and $\tau(x)$ is a polynomial with degree 1 [54].
For Hermite polynomials $\sigma(x) = 1$ and $\tau(x) = -2x$,
for Laguerre polynomials $\sigma(x) = x$ and $\tau(x) = \alpha + 1 - x$,
and for Jacobi polynomials $\sigma(x) = x^2 - 1$ and $\tau(x) = (\alpha + \beta + 2)x + \alpha - \beta$.
The semi-classical orthogonal polynomials satisfy the Pearson equation in (1.14) but with $\deg[\sigma(x)] > 2$ or $\deg\tau(x) \neq 1$. We need positive solution $w(x)$ such that

$$\sigma(a)w(a) = \sigma(b)w(b) = 0 \quad for \quad a, \ b \in \mathbb{R} \cup \{-\infty, +\infty\}. \tag{1.15}$$

A very important property which is satisfied by classical and semi-classical orthogonal polynomials known as *structure relation* has been established in [54], where the property is proved using the orthonormal polynomials but here we give the proof using the orthogonal polynomials on the real line in the general case.

Property 1.1. : If the weight function $w(x)$ satisfies the Pearson equation (1.14) and the conditions (1.15) satisfied, then

$$\sigma(x)p_n'(x) = \sum_{k=n-t}^{n+s-1} A_{n,k}p_k(x), \tag{1.16}$$

where $s = deg\sigma(x)$ and $t = max\{deg\tau(x),\ deg\sigma(x) - 1\}$.

Proof. Since $\deg\sigma(x) = s$ and deg $p_n(x) = n$, we have $\deg[\sigma p_n'(x)] = n + s - 1$. Then, the expansion of the polynomial $\sigma(x)p_n'(x)$ in terms of the orthogonal polynomials $p_k(x)$ with $0 \le k \le n + s - 1$ yields

$$\sigma(x)p_n'(x) = \sum_{k=0}^{n+s-1} A_{n,k}p_k(x)$$

where $A_{n,k}$ are Fourier coefficients which can be expressed as

$$A_{n,k} = \frac{1}{h_k}\int_a^b \sigma(x)p_n'(x)p_k(x)w(x)dx.$$

Applying the integration by parts with the boundary conditions (1.15), we have

$$\begin{aligned}
A_{n,k} &= \frac{1}{h_k}\int_a^b \sigma(x)p_n'(x)p_k(x)w(x)dx \\
&= -\frac{1}{h_k}\int_a^b p_n(x)[\sigma(x)w(x)p_k(x)]'dx \\
&= -\frac{1}{h_k}\int_a^b p_n(x)p_k(x)[\sigma(x)w(x)p_k]'dx - \frac{1}{h_k}\int_a^b p_n(x)p_k'(x)\sigma(x)w(x)dx \\
&= -\frac{1}{h_k}\int_a^b p_n(x)p_k(x)\tau(x)w(x)dx - \frac{1}{h_k}\int_a^b p_n(x)p_k'(x)\sigma(x)w(x)dx.
\end{aligned}$$

By orthogonality, for $k < n - deg\tau(x)$, we get

$$\frac{1}{h_k}\int_a^b p_n(x)p_k(x)\tau(x)w(x)dx = 0,$$

and for $k < n - (s-1)$, we still get

$$\frac{1}{h_k}\int_a^b p_n(x)p_k'(x)\sigma(x)w(x)dx = 0.$$

That is $A_{n,k} = 0$ for all $k < n - t$
with

$$t = max(deg\tau(x), s-1).$$

Hence,

$$\sigma(x)p_n'(x) = \sum_{k=n-t}^{n+s-1} A_{n,k}p_k(x).$$

□

Thus, as we have already seen, every sequence of orthogonal polynomials on the real line satisfies the three-term recurrence relation (1.3). Furthermore, classical and semi-classical orthogonal polynomials also satisfy the structure relation (1.16). Note that both relations should be compatible. Expressing the compatibility relations in terms of recurrence coefficients and the coefficients in the structure relation, we obtain a non linear recurrence relations for these coefficients. Therefore, when we get the expressions of recurrence coefficients, the sequence of orthogonal polynomials becomes completely determined. In what follows, we illustrate this using the Laguerre polynomials.

1.6 Some properties of Laguerre polynomials

In this section, we are interested in some properties which will be used in the last Chapter.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ with non-negative integer n and real $\alpha > -1$ are widely used in many problems of mathematical physics, quantum mechanics and chemistry. For examples:

- The radial part of the wave function of an electron in a coulomb potential is the product of Laguerre polynomial and an exponential with the variable scaled by a factor depending on the degree [19];
- the eigenfunctions for the Schrödinger operator associated with the hydrogen atom are described in terms of Laguerre polynomials [36].

etc.

1.6.1 Definition and first examples

Laguerre polynomials $L_n^{(\alpha)}(x)$ are orthogonal on $[0, \infty[$ with respect to the weight function $w(x) = x^\alpha e^{-x} \quad (\alpha > -1)$.
They can be defined by means of their Rodrigues formula [55]

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right), \quad n \in \mathbb{N}. \tag{1.17}$$

Using the Leibniz's rule

$$\frac{d^n}{dx^n}\left[f(x)g(x)\right] = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}f(x)\frac{d^{n-k}}{dx^{n-k}}g(x),$$

the formula (1.17) yields

$$\begin{aligned}
L_n^{(\alpha)}(x) &= \frac{x^{-\alpha}e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k e^{-x}}{dx^k}\frac{d^{n-k}}{dx^{n-k}}x^{n+\alpha}\\
&= \frac{x^{-\alpha}e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}(\alpha+n)(\alpha+n-1)(\alpha+n-2)...(\alpha+k+1)x^{\alpha+k}\\
&= \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n+\alpha)!}{(k+\alpha)!}x^k,
\end{aligned}$$

so that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \quad n \in \mathbb{N}. \tag{1.18}$$

This proves that $L_n^{(\alpha)}(x)$ is a polynomial of degree n with the leading coefficient $k_n = \frac{(-1)^n}{n!}$.

Using the formula (1.18), the first five Laguerre polynomials are :

$$L_0^{(\alpha)}(x) = 1,$$

$$L_1^{(\alpha)}(x) = \alpha + 1 - x,$$

$$L_2^{(\alpha)}(x) = \frac{(\alpha+2)(\alpha+1)}{2} - (\alpha+2)x + \frac{x^2}{2},$$

$$L_3^{(\alpha)}(x) = \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{6} - (\alpha+3)(\alpha+2)\frac{x}{2} + (\alpha+3)\frac{x^2}{2} - \frac{x^3}{6},$$

$$L_4^{(\alpha)}(x) = \frac{(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)}{24} - (\alpha+4)(\alpha+3)(\alpha+2)\frac{x}{6} + (\alpha+4)(\alpha+3)\frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{24}.$$

By plotting the first five polynomials on the interval $(0, 5)$ for $\alpha = 0$ we obtained the following graph

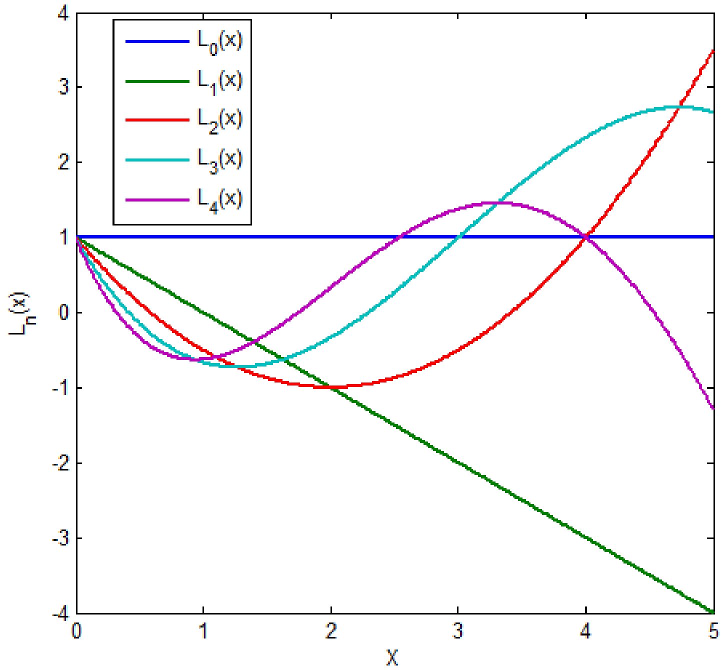

FIGURE 1.1: *The first five Laguerre polynomials on* $(0, 5)$ *for* $\alpha = 0$

1.6.2 Orthogonality

Theorem 1.5. *[5, 55] The Laguerre polynomials* $L_n^{(\alpha)}(x)$ $(\alpha > -1)$ *orthogonal on* $(0, \infty)$ *with respect to weight* $w(x) = x^\alpha e^{-x}$ *satisfy the following orthogonality property*

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{m,n}, \quad m, n \geq 0. \tag{1.19}$$

Proof. In view of (1.17), the orthogonality condition (1.1) implies

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{1}{n!} \int_0^\infty L_m^{(\alpha)}(x) \left(\frac{d^n}{dx^n} e^{-x} x^{n+\alpha} \right) dx.$$

By applying the integration by parts n times, the previous relation yields

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{(-1)^n}{n!} \int_0^\infty \left(\frac{d^n}{dx^n} L_m^{(\alpha)}(x) \right) e^{-x} x^{n+\alpha} dx = 0, \text{ for all } m < n.$$

Furthermore for $m = n$, we have

$$\int_0^\infty \left[L_n^{(\alpha)}(x)\right]^2 x^\alpha e^{-x} dx = \frac{1}{n!}\int_0^\infty e^{-x} x^{n+\alpha} dx = \frac{\Gamma(n+\alpha+1)}{n!}.$$

□

1.6.3 Three-term recurrence relation

Now, to illustrate how the compatibility between the relations (1.3) and (1.16) can lead to the explicit expressions of the recurrence coefficients $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 1}$, we take as example the case of Laguerre polynomials. Alternative way involving the generating function to find these coefficients can be found in [51].

Since for Laguerre polynomials we have $w(x) = x^\alpha e^{-x}$ $(\alpha > -1)$, $(a, b) = (0, \infty)$ and $\sigma(x) = x$, the corresponding Pearson equation is

$$[xw(x)]' = (\alpha + 1 - x)w(x). \tag{1.20}$$

Let us take the monic Laguerre polynomials. From (1.16), the corresponding structure relation is

$$xP_n'(x) = A_n P_n(x) + B_n P_{n-1}(x), \quad n \geq 1 \tag{1.21}$$

and taking derivatives in the three-term recurrence relation (1.5) yields

$$P_n(x) + xP_n'(x) = P_{n+1}'(x) + b_n P_n'(x) + c_n P_{n-1}'(x), \quad n \geq 1. \tag{1.22}$$

Multiplying this expression (1.22) by x and using the structure relation (1.21) to eliminate all derivatives, we get

$$xP_n(x) + x\left(A_n P_n(x) + B_n P_{n-1}(x)\right) =$$
$$A_{n+1}P_{n+1}(x) + B_{n+1}P_n(x) + b_n\left(A_n P_n(x) + B_n P_{n-1}(x)\right)$$
$$+c_n\left(A_{n-1}P_{n-1}(x) + B_{n-1}P_{n-2}(x)\right).$$

Now, we use the three-term recurrence relation (1.5) to replace xP_n and xP_{n-1} in the previous relation. Then, we get

$$(1+A_n)\left(P_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x)\right)+B_n\left(P_n(x)+b_{n-1}P_{n-1}(x)+c_{n-1}P_{n-2}(x)\right)$$

$$=A_{n+1}P_{n+1}(x)+B_{n+1}P_n(x)+b_n\left(A_nP_n(x)+B_nP_{n-1}(x)\right)+c_n\left(A_{n-1}P_{n-1}(x)+B_{n-1}P_{n-2}(x)\right).$$

Since the polynomials $\{P_{n+1}, P_n, P_{n-1}, P_{n-2}\}$ are linearly independent, the previous relation can be true if and only if the coefficients in front of similar polynomials on both sides of the previous equality are equal.
Therefore, equating the coefficients in front of P_{n+1}, P_n, P_{n-1} and P_{n-2}, we obtain

$$P_{n+1} \Rightarrow 1+A_n = A_{n+1} \tag{1.23}$$

$$P_n \Rightarrow b_n(1+A_n)+B_n = B_{n+1}+b_nA_n \tag{1.24}$$

$$P_{n-1} \Rightarrow c_n(1+A_n)+b_{n-1}B_n = b_nB_n+c_nA_{n-1} \tag{1.25}$$

$$P_{n-2} \Rightarrow c_{n-1}B_n = c_nB_{n-1}. \tag{1.26}$$

Now, we are going to solve for b_n and c_n the above system.
Thus, from (1.23), we get that $A_{n+1}-A_n = 1$ so that

$$A_n = n. \tag{1.27}$$

From (1.26), we then we have

$$\frac{B_n}{c_n} = \frac{B_{n-1}}{c_{n-1}},$$

so that this ratio is constant. Hence,

$$B_n = dc_n, \tag{1.28}$$

where d is a constant.
From (1.24), in view of (1.27) and (1.28), we obtain

$$(n+1)b_n + dc_n = dc_{n+1} + nb_n,$$

so that

$$b_n = d(c_{n+1}-c_n), \tag{1.29}$$

and from (1.25), again in view of (1.27) and (1.28), we get

$$(n+1)c_n + db_{n-1}c_n = db_nc_n + (n-1)c_n,$$

so that

$$d(b_n - b_{n-1}) = 2.$$

Summing this equation from 1 to n gives

$$b_n = \frac{2n}{d} + b_0. \tag{1.30}$$

Substituting this in (1.29), we obtain the following equation

$$d(c_{n+1} - c_n) = \frac{2n}{d} + b_0,$$

which after a telescopic sum of it gives

$$dc_n = \frac{n(n-1)}{d} + nb_0. \tag{1.31}$$

To find the explicit expressions of b_n and c_n, we still have to determine the constants b_0 and d which appear in (1.30) and (1.31).

For this, let us begin by computing b_0.
From (1.6), we get

$$b_0 = \frac{1}{h_0}\int_0^\infty xw(x)dx = \frac{m_1}{h_0},$$

and from (1.1), we get

$$h_0 = \int_0^\infty w(x)dx = m_0.$$

Hence,

$$\begin{aligned} b_0 &= \frac{m_1}{m_0} \\ &= \frac{\int_0^\infty x^{\alpha+1}e^{-x}dx}{\int_0^\infty x^\alpha e^{-x}dx} \\ &= \frac{(\Gamma+2)}{(\Gamma+1)} \\ &= \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \end{aligned}$$

so that

$$b_0 = \alpha + 1,$$

where we have used the gamma function defined by $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx$.
Now, to determine the constant d, we start by computing the Fourier coefficient B_1 in (1.21). Then,

$$\begin{aligned} B_1 &= \frac{1}{h_0}\int_0^\infty xP_1'(x)P_0(x)w(x)dx \\ &= -\frac{1}{h_0}\int_0^\infty P_1(x)(\alpha+1-x)w(x)dx \\ &= \frac{1}{h_0}\int_0^\infty xP_1(x)P_0w(x)dx, \end{aligned}$$

where we have used the integration by parts, then the Pearson equation (1.20) in the above. With the aid of (1.8), the previous equality yields $B_1 = c_1$.
Comparing this equality with that in (1.28) (for $n = 1$), we obtain

$$d = 1.$$

Hence, the recurrence coefficients of monic Laguerre polynomials are

$$b_n = 2n + \alpha + 1 \tag{1.32}$$

and

$$c_n = n(n + \alpha). \tag{1.33}$$

That is the three-term recurrence for monic Laguerre polynomials is

$$xP_n(x) = P_{n+1} + (2n+\alpha+1)P_n(x) + n(n+\alpha)P_{n-1}(x). \tag{1.34}$$

Note that these results are the same as those found by Van Assche for orthonormal Laguerre polynomials with this same method [54].
Note also that the usual Laguerre $L_n^{(\alpha)}(x)$ are not monic. To obtain them we can use the fact that $P_n(x) = L_n^{(\alpha)}/k_n$ with $k_n = (-1)^n/n!$.
Substituting this in (1.34), we obtain

$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \tag{1.35}$$

which is the three-term recurrence relation for Laguerre polynomials.

Chapter 2

Painlevé equations

2.1 A short history of Painlevé equations

The Painlevé equations $P_I - P_{VI}$ were first discovered in the late 19th and early 20th centuries $(1895-1910)$, in the investigations by Painlevé [47, 48] and Gambier [21] while studying the problem orginally posed by Picard in [49]. That problem is as follows :

Given $R(y', y, x)$ rational in y' and y and analytic in x, what are the second order ordinary differential equations of the form

$$y'' = R(y', y, x), \tag{2.1}$$

with the property that the singularities other than poles of any solution of (2.1) depend on the equation only and not on the constant of integration (i.e, with property that the general solution of (2.1) is free from movable branch points) ? This property is called Painlevé property.

By investigating this problem, Painlevé [47, 48] and Gambier [21] found 50 canonical equations of the form (2.1) that possess this Painlevé property, up to a Möbius (bilinear rational) transformation

$$Y(\zeta) = \frac{a(x)y + b(x)}{c(x)y + d(x)}, \qquad \zeta = \phi(x),$$

where $a(x), b(x), c(x), d(x)$ and $\phi(x)$ are locally analytic functions.

Moreover, they showed that 44 of them can be reduced to known equations (such as linear equations, the Weierstrass elliptic equations or Riccati equations) or the six new second-order nonlinear differential equations known as *Painlevé equations*.

According to the literature, for example in [48], as early as 1885, Méray, Briot and Weierstrass tackled the problem in a particular case, by studying the first order equations, i.e. $F(\frac{dy}{dx}, y) = 0$, which possess the Painlevé property. The transcendents thus defined merges with the elliptical functions and their degenerations. Thirty years later, Fuchs, generalizing the research of Briot and Bouquet, determined the differential equations whose possess the Painlevé property but with a brillant method of integration. Poincaré showed almost immediately that these equations are always reducible to quadratures and to linear differential equations of second order (see [48, p.3]). Furthermore, one notices that since the birth of the Painlevé equations, even the fundamental work of Painlevé, all has been done while thinking of the link between structure of singularity and integrability. As we can find in [26], this connection was first observed by Kowalevkaya [32, 33] while he was working on the equations of a spining top. The essence of these classic investigations was that whenever the movable singularities of a given system of differential equations are only poles, the system turns out to be integrable, in some sense. More recently, the integrability of Painlevé equations has been confirmed by the inverse scattering method, originally developed by Gardner et al.[22] in order to solve the Cauchy problem for the Korteweg-de Vries equation. Since, we can find others works about the connection between singularity structure and integrability, see for example, [1–3] for reference.

The Painlevé equations are important nonlinear special functions that they have significantly expanded the role started to play by special classical functions, such as, for example, the Bessel, Airy and hypergeometric functions. Nowadays, they appear in various situations such as orthogonal polynomials, partial differential such as nonlinear wave equations, fibre optics, random matrices theory, number theory, neutron scattering theory, quantum gravity, etc [16].

2.2 The six Painlevé differential equations

The six new nonlinear differential equations known as Painlevé equations are [54]

$$P_I \quad y'' = 6y^2 + x, \tag{2.2}$$

$$P_{II} \quad y'' = 2y^3 + xy + \alpha, \tag{2.3}$$

$$P_{III} \quad y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{2.4}$$

$$P_{IV} \quad y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{2.5}$$

$$\begin{aligned} P_V \quad y'' = &(\frac{1}{2y} + \frac{1}{y-1})(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{x} \\ &+ \frac{\delta y(y+1)}{y-1}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} P_{VI} \quad y'' = &\frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x})(y')^2 - (\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x})y' \\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right), \end{aligned} \tag{2.7}$$

where α, β, γ, δ are constants and $' = d/dx$.

The general solutions of these equations ($P_I - P_{VI}$) define new transcendent known as *Painlevé transcendents*.
More precisely, only the equations P_I, P_{II} and P_{III} were found by Painlevé, the others three were discovered by his student Gambier [21] who, continuing the revision of Painlevé computations, found that the latter had omitted one of the sub-case allowing to find the equations P_{IV}, P_V and P_{VI}.

Furthermore, with adequate transformations of the variable x and parameters $\alpha, \beta, \gamma$and δ, the Painlevé equations have the following coalescence cascade [28, p.125],[4, 14]

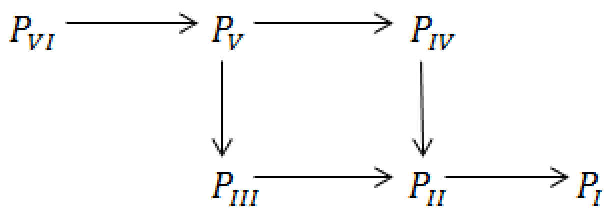

FIGURE 2.1: *Painlevé equations coalscence*

2.3 Some mathematical properties of Painlevé equations

2.3.1 Hamiltonian structure

The Painlevé equations P_J $(J = I, II, III, IV, V, VI)$ can be written as a Hamiltonian system

$$\frac{dq}{dx} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial H_J}{\partial q}, \tag{2.8}$$

for a suitable Hamiltonian function $H_J(q, p, x)$ [15, 41].
Furthermore, the function $\sigma_J(x) \equiv H_J(q, p, x)$ satisfies a second-order ordinary differential equation, known as *Painlevé* $\sigma-$*equation*, for which its solution is expressible in terms of the solution of associated Painlevé equation [41].
The six Painlevé $\sigma-$equations associated with P_J are respectively [16]:

$$S_I \quad \left(\frac{d^2\sigma}{dx^2}\right)^2 + 4\left(\frac{d\sigma}{dx}\right)^3 + 2x\frac{d\sigma}{d\sigma} - 2\sigma = 0, \tag{2.9}$$

$$S_{II} \quad \left(\frac{d^2\sigma}{dx^2}\right)^2 + 4\left(\frac{d\sigma}{dx}\right)^3 + 2\frac{d\sigma}{dx}\left(x\frac{d\sigma}{dx} - \sigma\right) - \frac{1}{4}\beta^2 = 0, \tag{2.10}$$

$$S_{III}\quad \left(x\frac{d^2\sigma}{dx^2}-\frac{d\sigma}{dx}\right)^2+\left[4\left(\frac{d\sigma}{dx}\right)^2-x^2\right]\left(x\frac{d\sigma}{dx}-2\sigma\right)+4x\theta_\infty\frac{d\sigma}{dx}-2\theta_0x^2=0, \tag{2.11}$$

$$S_{IV}\quad \left(\frac{d^2\sigma}{dx^2}\right)^2-4\left(x\frac{d\sigma}{dx}-\sigma\right)^2+4\frac{d\sigma}{dx}\left(\frac{d\sigma}{dx}+2\theta_0\right)\left(\frac{d\sigma}{dx}+2\theta_\infty\right)=0, \tag{2.12}$$

$$S_V\quad \left(x\frac{d^{2\sigma}}{dx^2}\right)^2-\left[2\left(\frac{d\sigma}{dx}\right)^2-x\frac{d\sigma}{dx}+\sigma\right]^2+4\prod_{j=1}^{4}\left(\frac{d\sigma}{dx}+\mathcal{K}_j\right)=0, \tag{2.13}$$

$$S_{VI}\quad \frac{d\sigma}{dx}\left[x(x-1)\frac{d^2\sigma}{dx^2}\right]^2+\left[\frac{d\sigma}{dx}\{2\sigma-(2x-1)\frac{d\sigma}{dx}\}+\prod_{j=1}^{4}\mathcal{K}_j\right]^2-\prod_{j=1}^{4}\left(\frac{d\sigma}{dx}+\mathcal{K}_j^2\right)=0, \tag{2.14}$$

where $\beta, \theta_0, \theta_\infty$ and $\mathcal{K}_j$ are arbitrary constants.

Example 2.1: [20, 44] For the Painlevé P_V , the Hamiltonian associated is defined by

$$xH_V(q,p,x)=q(q-1)^2p^2-\left[(\mathcal{K}_2-\mathcal{K}_1)(q-1)^2-2(\mathcal{K}_1+\mathcal{K}_2)q(q-1)+xq\right]p$$
$$+(\mathcal{K}_3-\mathcal{K}_1)(\mathcal{K}_4-\mathcal{K}_1)(q-1), \tag{2.15}$$

where the parameters $\mathcal{K}_i$ are constrained by

$$\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4=0. \tag{2.16}$$

Then , the Hamilton equations are

$$xq'=2q(q-1)^2p-\left[(\mathcal{K}_2-\mathcal{K}_1)(q-1)^2-2(\mathcal{K}_1+\mathcal{K}_2)q(q-1)+xq\right] \tag{2.17}$$

and

$$xp'=-(q-1)(3q-1)p^2+\left[2(\mathcal{K}_2-\mathcal{K}_1)(q-1)-2(\mathcal{K}_1+\mathcal{K}_2)(2q-1)+x\right]p$$
$$+(\mathcal{K}_3-\mathcal{K}_1)(\mathcal{K}_4-\mathcal{K}_1). \tag{2.18}$$

Eliminating p in the above Hamilton equations (2.17) and (2.18), we find that $q=y$ satisfies the Painlevé equation P_V in (2.6) with $\alpha=\frac{1}{2}(\mathcal{K}_3-\mathcal{K}_4)^2, \beta=-\frac{1}{2}(\mathcal{K}_2-\mathcal{K}_1)^2, \gamma=2(\mathcal{K}_1+\mathcal{K}_2)-1$ and $\delta=-\frac{1}{2}$. Note that the general Painlevé equation P_V in (2.6) with $\delta=0$ can be reduced to the case

with $\delta = -\frac{1}{2}$ by the mapping $x \mapsto \sqrt{-2\delta x}$. .
The function

$$\sigma = xH_V + (\mathcal{K}_3 - \mathcal{K}_1)(\mathcal{K}_4 - \mathcal{K}_1) - \mathcal{K}_1 x - 2\mathcal{K}_1^2 \tag{2.19}$$

defined as in (2.15) satisfies

$$(x\sigma'')^2 - \left[2(\sigma')^2 - x\sigma' + \sigma\right]^2 + 4\prod_{i=0}^{4}(\sigma' + \mathcal{K}_i) = 0, \tag{2.20}$$

which is the Painlevé σ-equation S_V in (2.12).

Indeed, from (2.15)-(2.18), we obtain

$$\sigma' = -qp - \mathcal{K}_1, \tag{2.21}$$

$$x\sigma'' = q(q^2-1)p^2 - \left[(\mathcal{K}_3 + \mathcal{K}_4 - 2\mathcal{K}_1)q^2 - \mathcal{K}_2 + \mathcal{K}_1\right] p + (\mathcal{K}_3 - \mathcal{K}_1)(\mathcal{K}_4 - \mathcal{K}_1)q. \tag{2.22}$$

These two equations (2.21) and (2.22) yield

$$x\sigma'' = q(\sigma' + \mathcal{K}_3)(\sigma' + \mathcal{K}_4) + p(\sigma' + \mathcal{K}_2). \tag{2.23}$$

Moreover, the equation (2.21) together with (2.15) and (2.19) give

$$2(\sigma')^2 - x\sigma' + \sigma = q(\sigma' + \mathcal{K}_3)(\sigma' + \mathcal{K}_4) - p(\sigma' + \mathcal{K}_2). \tag{2.24}$$

By substracting the square of equation (2.24) from that of equation (2.23), we get

$$(x\sigma'')^2 - \left[2(\sigma')^2 - x\sigma' + \sigma\right]^2 = 4qp(\sigma' + \mathcal{K}_2)(\sigma' + \mathcal{K}_3)(\sigma' + \mathcal{K}_4).$$

Using (2.21) in the previous equation, we obtain the desired equation (2.20).

Conversely [44], if σ is a solution of (2.20), then

$$q = \frac{1}{2(\sigma' + \mathcal{K}_3)(\sigma' + \mathcal{K}_4)}\left[x\sigma'' + \sigma - x\sigma' + 2(\sigma')^2\right] \tag{2.25}$$

and

$$p = \frac{1}{2(\sigma' + \mathcal{K}_2)}\left[x\sigma'' - \sigma + x\sigma' - 2(\sigma')^2\right], \tag{2.26}$$

are respectively solutions of (2.17) and (2.18).

For Hamiltonian structure for P_{II} and P_{IV} see Okamoto [43], for Hamiltonian sructure for P_{III} see Okamoto [45] and also Forrester and Witte [20] and for Hamiltonian structure for P_{VI} see Okamoto [42].

2.3.2 Bäcklund transformations

A Bäcklund transformation is defined as system of equations relating one solution of a given equation either to an other solution of the same equation but with different values of the parameters or to a solution of an other equation. Except the the Painlevé equation P_I, all the others Painlevé equations $P_{II} - P_{VI}$ possess Bäcklund transformations.
To illustrate this, mainly referring to [14], we take some examples for the Painlevé equation P_V given in (2.6) .
Let

$$y_i(x_i) = y(x_i, \alpha_i, \beta_i, \gamma_i, \delta_i), \quad i = 0, 1, 2 \quad ,$$

be solutions of P_V in (2.6) with

$$x_1 = -x_0, \;\; x_2 = x_0, \;\; (\alpha_1, \beta_1, \gamma_1, \delta_1) = (\alpha_0, \beta_0, -\gamma_0, \delta_0), \;\; (\alpha_2, \beta_2, \gamma_3, \delta_4) = (\beta_0, -\alpha_0, -\gamma_0, \delta_0).$$

Then, we have the following transformations

$$S_1 : \quad y_1(x_1) = y(x_0), \tag{2.27}$$

$$S_2 : y_2(x_2) = \frac{1}{y(x_0)}. \tag{2.28}$$

Furher, let

$$Y_0 = Y(x, \alpha_0, \beta_0, \gamma_0, -\frac{1}{2})$$

and

$$Y_1 = Y(x, \alpha_1, \beta_1, \gamma_1, -\frac{1}{2}),$$

be solutions of P_V(2.6), where

$$\alpha_1 = \frac{1}{8}\left[\gamma_0 + \varepsilon_1(1 - \varepsilon_3\sqrt{-2\beta_0} - \varepsilon_2\sqrt{2\alpha_0})\right]^2,$$

$$\beta_1 = -\frac{1}{8}\left[\gamma_0 - \varepsilon_1(1 - \varepsilon_3\sqrt{-2\beta_0} - \varepsilon_2\sqrt{2\alpha_0})\right]^2,$$

and $\varepsilon_i = \pm 1, \quad i = 1, 2, 3$, independently.
Also let

$$\phi = xY_0' - \varepsilon_2\sqrt{2\alpha_0}Y_0^2 + \varepsilon_3\sqrt{-2\beta_0} + (\varepsilon_2\sqrt{2\alpha_0} - \varepsilon_3\sqrt{-2\beta_0} + \varepsilon_1 x)Y_0,$$

and assume $\phi \neq 0$. Then, we get again the following transformations

$$\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}: \quad Y_1 = (\phi - 2\varepsilon_1 xY_0)/\phi, \tag{2.29}$$

provided that the numerator on the right-hand side does not vanish. The fact that, $\varepsilon_i = \pm 1, \quad i = 1, 2, 3$, independently, implies that there are eight distinct thransformations of type $\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$.
For more details see [24], also [26, §39] and [23].

2.3.3 Rational solutions

The general solutions of Painlevé equations are said to be new transcendental functions in the sense that they cannot be expressed in terms of known functions. However, the Painlevé equations $P_{II} - P_{VI}$ can have rational solutions and solutions expressible in terms of known special functions for particular values of the parameters which are generated using the Bäcklund transformations.

For instance, for the Painlevé equation P_V in (2.6), some simple rational solutions are :

$$y(x, \frac{1}{2}, -\frac{1}{2}\mu^2, \mathcal{K}(2-\mu), -\frac{1}{2}\mathcal{K}^2) = \mathcal{K}x + \mu, \tag{2.30}$$

$$y(x, \frac{1}{2}, \mathcal{K}^2\mu, 2\mathcal{K}\mu, \mu) = \frac{\mathcal{K}}{\mathcal{K} + x}, \tag{2.31}$$

$$y(x, \frac{1}{8}, -\frac{1}{8}, -\mathcal{K}\mu, \mu) = \frac{\mathcal{K} + x}{\mathcal{K} - x}, \tag{2.32}$$

where $\mathcal{K}$ and μ are arbitrary constants.
In the general, we take the case with $\delta \neq 0$ and we set $\delta = -\frac{1}{2}$ without loss the generality. The case $\delta = 0$ can be reduced to P_{III} (see [26, §38]). The rational solutions of P_V are classified by mean of the following theorem [14, Thm.5.13]:

Theorem 2.1. *The Painlevé equation P_V in (2.6), with $\delta = -\frac{1}{2}$, has rational solutions if and only if one of the following holds for $(m, n) \in \mathbb{Z}^2$.*

(*i*) $\alpha = \frac{1}{2}(m \pm 1)^2$ *and* $\beta = -\frac{1}{2}n^2$*, where* $n > 0$*,* $m + n$ *is odd and* $\alpha \neq 0$ *when* $|m| < n$*,*

(*ii*) $\alpha = \frac{1}{2}n^2$ *and* $\beta = -\frac{1}{2}(m \pm 1)^2$*, where* $n > 0$*,* $m + n$ *is odd,* $\beta \neq 0$ *when* $|m| < n$*,*

(*iii*) $\alpha = \frac{1}{2}a^2$*,* $\beta = -\frac{1}{2}(a + n)^2$ *and* $\gamma = m$*, where* $m + n$ *is even and* b *arbitrary,*

(*iv*) $\alpha = \frac{1}{2}(b + n)^2$*,* $\beta = -\frac{1}{2}b^2$ *and* $\gamma = m$*, where* $m + n$ *is even and* b *arbitrary,*

(*v*) $\alpha = \frac{1}{8}(2m + 1)^2$ *and* $\beta = -\frac{1}{8}(2n + 1)^2$*.*

These rational solutions have the form

$$y(x) = \lambda x + \mu + P_{n-1}(x)/Q_n(x), \tag{2.33}$$

with λ *and* μ *constants, and* $P_{n-1}(x)$ *and* $Q_n(x)$ *polynomials of degree* $n - 1$ *and* n*, respectively, also with no common roots.*

Proof. See [26, §40] also [25]. □

The cases (i) and (ii) are the special case of the solutions of P_V expressible in terms of confluent hyper-geometric functions , which will be described in the subsection 2.3.4, when the confluent hyper-geometric functions are Laguerre polynomials (see also [14, Thm.7.6]).
Following the works of Umemura [52] and Naoumi-Yamada [40] who found sets of polynomials (Umemura polynomials) with which rational solutions of P_V can be constructed, Masuda ,Ohta and Kajiwara [35] generalised these results and derived special polynomials associated with all rational solutions in cases (iii), (iv) and (v). These generalized results are given in the following theorem [14, 54].

Theorem 2.2. *Suppose that* $U_{m,n}(x, \mu)$ *satisfies*

$$U_{m+1,n}U_{m-1,n} = 8x[U_{m,n}U''_{m,n} - (U'_{m,n})^2] + 8U_{m,n}U'_{m,n}$$
$$+ (x + 2\mu - 2 + 2n - 6m)U^2_{m,n}, \tag{2.34}$$

$$U_{m,n+1}U_{m,n-1} = 8x[U_{m,n}U''_{m,n} - (U'_{m,n})^2] + 8U_{m,n}U'_{m,n}$$
$$+ (x - 2\mu - 2 + 2m - 6n)U^2_{m,n}, \tag{2.35}$$

with $U_{-1,1}(x,\mu) = U_{-1,0}(x,\mu) = U_{0,-1}(x,\mu) = U_{0,0}(x,\mu) = 1$.
Then $U_{m,n}$ *is a polynomial of degree* $\frac{1}{2}m(m+1) + \frac{1}{2}n(n+1)$ *and*

$$y(x) = \frac{U_{m,n-1}(x,\mu)U_{m-1,n}(x,\mu)}{U_{m-1,n}(x,\mu-2)U_{m,n-1}(x,\mu+2)} \tag{2.36}$$

is rational solutions of P_V *in* (2.6) *for*

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{1}{8}\mu^2, -(\mu-2m+2n)^2, -m-n, -\frac{1}{2}\right),$$

and

$$y(x) = -\frac{U_{m,n-1}(x,\mu+1)U_{m,n+1}(x,\mu-1)}{U_{m-1,n}(x,\mu-1)U_{m+1,n}(x,\mu+1)}, \tag{2.37}$$

for

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{1}{8}(2m+1)^2, -\frac{1}{2}(2n+1)^2, m-n-\mu, -\frac{1}{2}\right).$$

Proof. See Masuda, Ohta and Kajiwara [35], also Clarkson [13]. □

The polynomials $U_{m,n}$ are generalized Umemura polynomials. The original Umemura polynomials are $U_n(x,\mu) = U_{0,n}(x,\mu)$ and $U_{-n}(x,\mu) = U_{n-1,0}(-x,\mu-1)$ for $n \geq 0$. The rational solutions of P_V in (2.6) possess determinant representation with entries expressed in terms of Laguerre polynomials. See again Masuda, Ohta and Kajiwara [35]. Furthermore, Naoumi and Yamada [40] gave a determinantal representation of these rational solutions of P_V in terms of Shur functions.

For rational solutions for P_{II} see [15, 30], for P_{III} see [29, 54], for P_{IV} see [14, 43] and for P_{VI} see [14, 37].

2.3.4 Special function solutions

In this subsection, we mainly refer to [14, 54] and reference therein.

For some special values of the parameters, except P_I, all the Painlevé equations $P_{II} - P_{VI}$ have special solutions expressible in terms of classical special functions. Each of the special function solutions is generated through a Riccati equation of the form

$$y'(x) = p_2(x)y^2 + p_1(x)y + p_0(x), \tag{2.38}$$

with $p_0(x), p_1(x)$ and $p_2(x)$ rational functions. The special function solutions which are often referred to as " one-parameter solutions " (for they depend on one arbitrary constant of integration of the Ricatti equation), are generated using the above equation (2.38) and the Bäcklund transformations. Moreover, the special function solutions can also appear in the form of determinants as we have seen in the case of rational solutions. The results found on these special function solutions are summarized in the following table :

	$p_2(x)$	$p_1(x)$	$p_0(x)$	Condition on parameters	Special function
P_{II}	ε_1	0	$\frac{1}{2}\varepsilon_1 x$	$\alpha = \frac{1}{2}\varepsilon_1$	Airy functions $Ai(x), Bi(x)$
P_{III}	ε_1	$\frac{\varepsilon_1\alpha-1}{x}$	ε_2	$\varepsilon_1\alpha + \varepsilon_2\beta = 2$, $\gamma = 1, \delta = -1$	Bessel functions $J_\nu(x), Y_\nu(x)$ or modified Bessel functions $I_\nu(x), K_\nu(x)$
P_{IV}	ε_1	$2\varepsilon_1 x$	$-2(1+\varepsilon_1\alpha)$	$\beta = -2(1+\varepsilon_1\alpha)^2$	Weber-Hermite (parabolic cylinder) function $U(a,x)$
P_V	$\frac{a}{x}$	$\varepsilon_3 + \frac{b-a}{x}$	$-\frac{b}{x}$	$a+b+\varepsilon_3\gamma = 2n+1$, $\delta = -\frac{1}{2}$, $a = \varepsilon_1\sqrt{2\alpha}$, $b = \varepsilon_2\sqrt{-2\beta}$	Kummer functions $M(a,b,x), U(a,b,x)$
P_{VI}	$\frac{a}{x(x-1)}$	$b+c-\frac{a+c}{x}$	$-\frac{b}{x-1}$	$a+b+c+d = 2n+1$, $a = \varepsilon_1\sqrt{2\alpha}$, $b = \varepsilon_2\sqrt{-2\beta}$, $c = \varepsilon_3\sqrt{2\gamma}$, $d = \varepsilon_4\sqrt{1-2\gamma}$	hypergeometric function ${}_2F_1(a,b,c,x)$

TABLE 2.1: *Special function solutions of $P_{II} - P_{VI}$.*

Note that there are others mathematical properties of Painlevé equations such as other elementary solutions, integral equations, differential equations, isomonodromy problems, asymptotic expansion, etc, but we prefer now not to develop them here . For more details, see for examples, [14, 46] and references therein.

2.4 Discrete Painlevé equations

In this section we mainly refer to [18, 53, 54].

The discrete Painlevé equations are nonlinear discrete equations for which the continuous limit is one of the Painlevé differential equations. They appeared more recently and since their discovery, they have turned out have a wide range of properties similar to those of their continuous counterparts. One of the most pertinent of these properties is the *singularity confinement* which is considered as the discrete version of the Painlevé property.
Consider the recurrence relation $x_n = f(x_{n-2}, x_{n-1}, n) =$ with f a rational function. Let n_0 be an index such that $(x_{n_0-2}, x_{n_0-1}, n_0)$ gives a singularity of f , so that x_{n_0} is not defined. Then singularity confinement means that there is an integer p such that the singularity is confined to elements $x_{n_0}, x_{n_0+1}, ..., x_{n_0+p}$ but x_{n_0+p+1} is again defined and it depends on x_{n_0-1}. The usual Painlevé property for differential is that the only movable singularities of solutions of Painlevé equations are poles which are isolated singularities, hence a discrete version of poles as singularities is to require that singularities of a discrete Painlevé equation are confined.
For instance [53], consider the following equation which is a discrete Painlevé equation I.

$$x_n(x_{n+1} + x_n + x_{n-1}) = n,$$

which can be rewritten as

$$x_{n+1} = \frac{n}{x_n} - x_n - x_{n-1}.$$

If $x_n = 0$, then for x_{n+1} we have a singularity which gives $\pm\infty$. For x_{n+2} we then obtain $\mp\infty$ and for x_{n+3} we have $(\pm\infty) + (\mp\infty)$. A more careful analysis gives that for x_n near 0

$$\begin{aligned}
x_n &= \epsilon, \\
x_{n+1} &= \frac{n}{\epsilon} - x_{n-1} - \epsilon, \\
x_{n+2} &= -\frac{n}{\epsilon} + x_{n-1} + \frac{n+1}{n}\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+3} &= -\frac{n+3}{n}\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+4} &= \frac{n}{n+3}x_{n-1} + \mathcal{O}(\epsilon).
\end{aligned}$$

As $\epsilon \to 0$, we see that the indeterminate form for x_{n+3} becomes 0, but it does not give a new singularity for x_{n+4}. So the singularity is confined to x_{n+1}, x_{n+2} and x_{n+3}.

Compared to the six differential Painlevé equations, the discrete Painlevé equations are very numerous and the construction of their canonical list is more complicated. So the list of standard Painlevé equations grew historically as researchers discovered the equations. We give here some very important discrete Painlevé equations [18, 53, 54]

$$d\text{-}P_I \quad x_{n+1} + x_n + x_{n-1} = \frac{z_n + a(-1)^n}{x_n} + b, \tag{2.39}$$

$$d\text{-}P_{II} \quad x_{n+1} + x_{n-1} = \frac{x_n z_n + a}{1 - x_n^2}, \tag{2.40}$$

$$d\text{-}P_{IV} \quad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2}, \tag{2.41}$$

$$\begin{aligned} d\text{-}P_V \quad & \frac{(x_{n+1} + x_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_{n+1} + x_n)}(x_n + x_{n-1}) \\ & = \frac{[(x_n - z_n)^2 - a^2][(x_n - z_n)^2 - b^2]}{(x_n - c^2)(x_n - d^2)}, \end{aligned} \tag{2.42}$$

where $z_n = \alpha n + \beta$ and a, b, c, d constants.

When we take a good look at the list above, we notice that equations $d\text{-}P_{III}$ and $d\text{-}P_{VI}$ do not appear in the list. This is because, in the list above, x_n and x_{n+1} (and x_n and x_{n-1}) appear in additive form . They appear in other form multiplicative of discrete Painlevé equations known as q-discrete Painlevé equations as follows

$$q\text{-}P_{III} \quad x_{n+1}x_{n-1} = \frac{(x_n - aq_n)(x_n - bq_n)}{(1 - cx_n)(1 - x_n/c)}, \tag{2.43}$$

$$q\text{-}P_V \; (x_{n+1}x_n - 1)(x_{n+1}x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - cx_nq_n)(1 - x_nq_n/c)}, \tag{2.44}$$

$$\begin{aligned} q\text{-}P_{VI} \quad & \frac{(x_nx_{n+1} - q_nq_{n+1})(x_nx_{n-1} - q_nq_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} \\ & = \frac{(x_n - aq_n)(x_n - q_n/a)(x_n - bq_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}, \end{aligned} \tag{2.45}$$

where $q_n = q_0q^n$ and a, b, c, d are constants.

There are also others forms of discrete Painlevé equations such as asymmetric discrete Painlevé equations, alternative discrete Painlevé equations. For more examples, see our appendix.

Chapter 3

Deformed Laguerre weight function and Painlevé equation V

3.1 Introduction

This Chapter concerns the main contribution of this thesis. We investigate the Hankel determinant degenerated by the deformed Laguerre weight $w(x,t) = x^{\alpha}(x+1)^{\beta}e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0$. Remark that if $\beta = 0$ and $t = 1$, this weight function is reduced to the classical Laguerre weight. By applying the ladder operators approach that we will introduce in section 3.3, we obtain two auxiliary quantities $R_n(t)$ and $r_n(t)$ in which are expressed the recurrence coefficients of the monic orthogonal polynomials with respect to the above weight and we show that they satisfy the coupled Riccati equations (see Lemma 3.5), from which we find that $R_n(t)$, up to a certain linear fractional transformation, satisfies a particular fifth Painlevé equation $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$ (see Theorem 3.6).

Basor and Chen [7] had studied an other deformed Laguerre weight function $w(x,t) = x^{\alpha}(x+t)^{\lambda}e^{-x},\ \alpha > -1,\ t > 0,\ x > 0$ which is close to ours. They also found a particular Painlevé V but up to a linear fractional transformation of R_n different from ours.

3.2 Preliminaries

We start with a recall of some elementary facts seen about the orthogonal polynomials in Chapter 1. In what follows, we will use the monic polynomials P_n with respect to the weight $w(x,t) = x^{\alpha}(x+1)^{\beta}e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0$ over the interval $[0,\infty)$. The orthogonality condition in (1.1) becomes

$$\int_0^{\infty} P_m(x,t)P_n(x,t)w(x)dx = h_n(t)\delta_{m,n}, \tag{3.1}$$

where $h_n(t) > 0, m, n = 0; 1; 2; \ldots\quad,$

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and

$$w(x,t) = x^{\alpha}(x+1)^{\beta}e^{-tx},\ x \in [0,\infty[,\ \alpha > -1,\ \beta \in \mathbb{R},\ t > 0.$$

We also recall that the three terms recurrence relation is (1.4)

$$xP_n(x,t) = P_{n+1}(x,t) + b_n(t)P_n(x,t) + c_n(t)P_{n-1}(x,t), \tag{3.2}$$

where

$$P_n(x,t) = x^n + p(n,t)x^{n-1} + \ldots, \tag{3.3}$$

with $P_0(x,t) = 1,\ c_0(t)P_{-1}(x,t) = 0.$
An easy consequence of (3.1), (3.2) and (3.3) gives

$$b_n(t) = p(n,t) - p(n+1,t) \tag{3.4}$$

and (1.7)

$$c_n(t) = \frac{h_n(t)}{h_{n-1}(t)}. \tag{3.5}$$

A telescopic sum of (3.4) yields

$$\sum_{j=0}^{n-1} b_j(t) = -p(n,t). \tag{3.6}$$

We will also consider the Hankel determinant defined as in (1.9) but generated by the above deformed Laguerre weight, namely

$$D_n(t) := det\left(\int_0^{\infty} x^{i+j} w(x,t)dx\right)_{i,j=0}^{n-1}.$$

The moments in (1.2) can be computed as follows:

$$m_k(t) = (-1)^k \frac{d^k}{dt^k} m_0(t), \quad k = 0,\ 1,\ 2, \dots \quad ,$$

where

$$m_0(t) = \Gamma(\alpha+1)U(\alpha+1, \alpha+\beta+2, t)$$

and $U(a,b,z)$ is the Kummer function which can be represented as follows:

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} x^{a-1}(x+1)^{b-a-1} e^{-zx} dx.$$

Furthermore, it is well known that [27]

$$D_n(t) = \prod_{j=0}^{n-1} h_j(t). \tag{3.7}$$

Note that in the following discussions, for convenience, we will suppress the t dependence in $w(x,t)$, $P_n(x,t)$, $h_n(t)$, $b_n(t)$ and $c_n(t)$ unless it is needed.

3.3 Ladder operators and its supplementary conditions

The ladder operators approach is one of the useful techniques to find the recurrence coefficients for orthogonal polynomials and in study of Hankel determinants which play a fundamental role in random matrix theory. For some special weights, it was shown that there is often a Painlevé equation associated to the generated Hankel determinant, see, for examples [6–9, 38, 58]. These ladder operators have been known to various authors but Magnus in [34] noted that such operators were known to Laguerre.

For a convenient form of using operators and its supplementary conditions for orthogonal polynomials, in the following theorems of this section, we refer to [11, 39] and [10].

Theorem 3.1. *Let $w(x)$ be a smooth weight function defined on (a, b) such that $w(a) = w(b) = 0$. The monic orthogonal polynomials with respect to $w(x)$ satisfy the lowering operator equation*

$$\left(\frac{d}{dz} + B_n(z)\right) P_n(z) = c_n A_n(z) P_{n-1}(z), \tag{3.8}$$

where

$$A_n(z) := \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(x)}{z - x} P_n^2(x) w(x) dx, \tag{3.9}$$

$$B_n(z) := \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(x)}{z - x} P_n(x) P_{n-1}(x) w(x) dx \tag{3.10}$$

and $v(z) := -\ln w(z)$.

Proof. Since $P_n'(z)$ is a polynomial of degree $n - 1$, it can be expanded as

$$P_n'(z) = \sum_{k=0}^{n-1} \alpha_{n,k} P_k(z). \tag{3.11}$$

Using the orthogonality condition (3.1) and integration by parts, we have

$$\begin{aligned} \alpha_{n,k} &= \frac{1}{h_k} \int_a^b P_n'(x) P_k(x) w(x) dx \\ &= -\frac{1}{h_k} \int_a^b P_n(x) \left(P_k(x) w(x)\right)' dx \\ &= \frac{1}{h_k} \int_a^b P_n(x) P_k(x) v'(x) w(x) dx, \end{aligned}$$

where we have used $v(z) = -\ln w(z)$.

Substituting the previous equality into (3.11), we have

$$\begin{aligned} P_n'(z) &= \sum_{k=0}^{n-1}\left[\frac{\int_a^b P_n(x)P_k(x)v'(x)w(x)dx}{h_k}\right]P_k(z) \\ &= \int_a^b \left[\sum_{k=0}^{n-1}\frac{P_k(z)P_k(x)}{h_k}\right]P_n(x)v'(x)w(x)dx. \end{aligned}$$

If $v'(x)$ is replaced by $v'(z)$, we remark that the right-hand side of the above equality vanishes. Hence, it can be rewritten as

$$P_n'(z) = \int_a^b \left[\sum_{k=0}^{n-1}\frac{P_k(z)P_k(x)}{h_k}\right][v'(z)-v'(x)]P_n(x)w(x)dx.$$

Using the Christoffel-Darboux formula

$$\sum_{k=0}^{n-1}\frac{P_k(z)P_k(x)}{h_k} = \frac{P_n(z)P_{n-1}(x)-P_n(x)P_{n-1}(z)}{h_{n-1}(z-x)},$$

we obtain

$$\begin{aligned} P_n'(z) &= \int_a^b \left[\frac{P_n(z)P_{n-1}(x)-P_n(x)P_{n-1}(z)}{h_{n-1}(z-x)}\right][v'(z)-v'(x)]P_n(x)w(x)dx \\ &= \left(\frac{1}{h_{n-1}}\int_a^b \frac{v'(z)-v'(x)}{z-x}P_n(x)P_{n-1}(x)w(x)dx\right)P_n(z) \\ &\quad -\left(\frac{1}{h_n}\int_a^b \frac{v'(z)-v'(x)}{z-x}P_n^2(x)w(x)dx\right)P_{n-1}(z), \end{aligned}$$

which is, in view of (3.5), the equation (3.8). □

Theorem 3.2. *The functions* $A_n(z)$ *and* $B_n(z)$ *defined as in* (3.9) *and* (3.10) *satisfy the following equations :*

$$B_{n+1}(z)+B_n(z) = (z-b_n)A_n(z)-v'(z), \tag{S_1}$$

$$c_{n+1}A_{n+1}-c_nA_{n-1} = 1+(z-b_n)[B_{n+1}(z)-B_n(z)], \tag{S_2}$$

$$B_n^2(z)+v'(z)B_n(z)+\sum_{j=0}^{n-1}A_j(z) = c_nA_n(z)A_{n-1}(z). \tag{S_2'}$$

Proof. From the definition of $B_n(z)$ in (3.10), we obtain

$$B_{n+1}(z) + B_n(z) = \int_a^b \frac{v'(z) - v'(x)}{z - x} \left(\frac{P_{n+1}(x)}{h_n} + \frac{P_{n-1}(x)}{h_{n-1}} \right) P_n(x)w(x)dx \quad (3.12)$$

and from (3.2) in view of (3.5), we obtain

$$\frac{P_{n+1}(x)}{h_n} + \frac{P_{n-1}(x)}{h_{n-1}} = \frac{(x - b_n)}{h_n} P_n(x).$$

Substituting this previous equation in (3.12) gives

$$B_{n+1}(z) + B_n(z) = \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(x)}{z - x} (x - b_n) P_n^2(x) w(x) dx.$$

In view of the definition $A_n(z)$, it follows that

$$\begin{aligned} B_{n+1}(z) + B_n(z) - (z - b_n)A_n(z) &= -\frac{1}{h_n} \int_a^b (v'(z) - v'(x)) P_n^2(x) w(x) dx \\ &= -v'(z) + \frac{1}{h_n} \int_a^b v'(x) P_n^2(x) w(x) dx \\ &= -v'(z) + \frac{2}{h_n} \int_a^b P_n(x) P_n'(x) w(x) dx, \end{aligned}$$

so that

$$B_{n+1}(z) + B_n(z) - (z - b_n)A_n(z) = -v'(z),$$

which is (S_1).

To prove (S_2) , similarly we start with the definition of $B_n(z)$ in (3.10). Thus, we have

$$\begin{aligned} (z-b_n)[B_{n+1}(z)-B_n(z)] &= \int_a^b (v'(z) - v'(x)) \left(\frac{P_{n+1}(x)}{h_n} - \frac{P_{n-1}(x)}{h_{n-1}} \right) P_n(x)w(x)dx \\ &+ \int_a^b (v'(z) - v'(x)) (x - b_n) \left(\frac{P_{n+1}(x)}{h_n} - \frac{P_{n-1}(x)}{h_{n-1}} \right) P_n(x)w(x)dx. \end{aligned} \quad (3.13)$$

where we have used $z - b_n = (z - x) + (x - b_n)$.
From (3.2), we obtain

$$(x - b_n)P_n(x) = P_{n+1}(x) + c_n P_n(x).$$

Substituting this in (3.13) in view of (3.5), we obtain

$$\begin{aligned}(z-b_n)[B_{n+1}(z)-B_n(z)] &= \int_a^b (v'(z)-v'(x))\left(\frac{P_{n+1}(x)}{h_n}-\frac{P_{n-1}(x)}{h_{n-1}}\right)P_n(x)w(x)dx \\ &+\int_a^b \frac{v'(z)-v'(x)}{z-x}\left(\frac{P_{n+1}^2(x)}{h_n}-c_n\frac{P_{n-1}^2(x)}{h_{n-1}}\right)w(x)dx.\end{aligned}$$

Using (3.9), it follows that

$$c_{n+1}A_{n+1}-c_nA_{n-1}-(z-b_n)[B_{n+1}(z)-B_n(z)]$$

$$=\int_a^b (v'(z)-v'(x))\left(\frac{P_n(x)P_{n-1}(x)}{h_{n-1}}-\frac{P_{n+1}(x)P_n(x)}{h_n}\right)w(x)dx.$$

Integration by parts yields

$$c_{n+1}A_{n+1}-c_nA_{n-1}-(z-b_n)[B_{n+1}(z)-B_n(z)]$$

$$\begin{aligned} &= -\frac{1}{h_{n-1}}\int_a^b P_n'(x)P_{n-1}(x)w(x)dx \\ &\quad +\frac{1}{h_{n-1}}\int_a^b P_{n+1}'(x)P_n(x)w(x)dx \\ &= -n+(n+1),\end{aligned}$$

so that

$$c_{n+1}A_{n+1}-c_nA_{n-1}-(z-b_n)=1,$$

which the desired formula (S_2).

The equation (S_2') is proved by combination of (S_1) and (S_2). Multiplying (S_2) by $A_n(z)$ on both sides, we find

$$c_{n+1}A_{n+1}A_n(z)-c_nA_n(z)A_{n-1}(z)=A_n(z)+[B_{n+1}^2(z)-B_n^2(z)]+v'(z)[B_{n+1}(z)-B_n(z)].$$

A telescopic sum of this equation using $c_0=0$ and $B_0(z)=0$ gives

$$c_nA_n(z)A_{n-1}(z)=\sum_{j=0}^{n-1}A_j(z)+B_n^2(z)+v'(z)B_n(z).$$

This completes the proof. $\square$

Theorem 3.3. *The monic orthogonal polynomials $P_n(z)$ satisfy the raising operator equation*

$$\left(\frac{d}{dz} - B_n(z) - v'(z)\right) P_{n-1}(z) = -A_n(z)P_n(z), \tag{3.14}$$

where $A_n(z)$ and $B_n(z)$ are defined as in (3.9) and (3.10).

Proof. From (3.8), the replacing of n by $n-1$, we find

$$P'_{n-1}(z) = c_{n-1}A_{n-1}(z)P_{n-2}(z) - B_n(z)P_{n-1}(z) \tag{3.15}$$

and from (3.2), we have

$$c_{n-1}P_{n-2}(z) = (z - b_{n-1})P_{n-1}(z) - P_n(z).$$

Substituting this into (3.15), we obtain

$$P'_{n-1}(z) = [(z - b_n)A_{n-1}(z) - B_{n-1}(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z).$$

Use of (S_1) gives

$$P'_{n-1}(z) = (B_n(z) + v'(z))\, P_{n-1}(z) - A_{n-1}(z)P_n(z).$$

which is the desired equation (3.14). □

Remark 3.1. If w does not vanish at the end points of interval of orthogonality (a, b), additional terms would have to be included in the definitions of $A_{n(z)}$ and $A_{n(z)}$ (see [10, 11]). For our problem we assumed that $w(0) = w(\infty) = 0$.

3.4 Ladder operators and Deformed Laguerre weight

In this section, we apply the ladder operators and its supplementary conditions introduced in the above to our problem.

For our problem, we have

$$w(x,t) = x^{\alpha}(x+1)^{\beta}e^{-tx},\ \ x \in [0, \infty[,\ \ \alpha > -1,\ \ \beta \in \mathbb{R},\ \ t > 0 \quad ,$$

$$\begin{aligned} v(z) &= -\ln z^{\alpha}(z+1)^{\beta}e^{-tz} \\ &= -[\alpha \ln z + \beta \ln(z+1) - tz], \end{aligned}$$

so that

$$v(z) = tz - \alpha \ln z - \beta \ln(z+1).$$

Hence,

$$v'(z) = t - \frac{\alpha}{z} - \frac{\beta}{z+1}$$

and

$$\begin{aligned} \frac{v'(z) - v'(x)}{z - x} &= \frac{1}{z-x}\left(t - \frac{\alpha}{z} - \frac{\beta}{z+1} - t + \frac{\alpha}{x} + \frac{\beta}{x+1}\right) \\ &= \frac{\alpha}{zx} + \frac{\beta}{(z+1)(x+1)}. \end{aligned}$$

Since the right-hand side of the above formula is rational in z, it is easily seen that both $A_n(z)$ and $B_n(z)$ are also rational in z from their definitions in (3.9) and (3.10). More precisely, we have the following proposition.

Proposition 3.1 *Four our problem, we have*

$$A_n(z) = \frac{R_n}{z} + \frac{t - R_n}{z+1}, \tag{3.16}$$

$$B_n(z) = \frac{r_n}{z} - \frac{n + r_n}{z+1}, \tag{3.17}$$

where

$$R_n := \frac{\alpha}{h_n}\int_0^{\infty} \frac{P_n^2(x)w(x)}{x}dx, \tag{3.18}$$

$$r_n := \frac{\alpha}{h_{n-1}}\int_0^{\infty} \frac{P_n(x)P_{n-1}(x)w(x)}{x}dx. \tag{3.19}$$

Proof. Let us begin by the proof of (3.16).
According to the definition of $A_n(z)$ in (3.9), we have

$$\begin{aligned} A_n(z) &= \frac{1}{h_n}\int_0^{\infty}\left[\frac{\alpha}{zx} + \frac{\beta}{(z+1)(x+1)}\right]P_n^2(x)w(x)dx \\ &= \frac{\alpha}{zh_n}\int_0^{\infty}\frac{P_n^2(x)w(x)}{x}dx + \frac{\beta}{(z+1)h_n}\int_0^{\infty}\frac{P_n^2(x)w(x)}{x+1}dx \end{aligned}$$

These above two integrals have a simple relation via integration by parts. Indeed,

$$\begin{aligned}
\frac{\beta}{h_n}\int_0^\infty \frac{P_n^2(x)w(x)}{x+1}dx &= \frac{\beta}{h_n}\int_0^\infty \frac{P_n^2(x)x^\alpha(x+1)^\beta e^{-tx}}{x+1}dx \\
&= \frac{1}{h_n}\int_0^\infty x^\alpha e^{-tx} d(x+1)^\beta \\
&= -\frac{1}{h_n}\int_0^\infty \left(P_n^2(x)x^\alpha e^{-tx}\right)'(x+1)^\beta dx \\
&= \underbrace{\frac{-2}{h_n}\int_0^\infty P_n(x)P_n'(x)w(x)dx}_{0} - \frac{\alpha}{h_n}\int_0^\infty \frac{P_n^2(x)w(x)}{x}dx \\
&\quad \underbrace{+\frac{t}{h_n}\int_0^\infty P_n^2(x)w(x)dx}_{t} \\
&= t - \frac{\alpha}{h_n}\int_0^\infty \frac{P_n^2(x)w(x)}{x}dx,
\end{aligned}$$

so that

$$\frac{\beta}{h_n}\int_0^\infty \frac{P_n^2(x)w(x)}{x+1}dx = t - R_n.$$

Hence, this yields

$$A_n(z) = \frac{R_n}{z} + \frac{t-R_n}{z+1},$$

which is the formula (3.16).

Now, similarly we give the proof of (3.17).
According to the definition of $B_n(z)$ in(3.10), we have

$$\begin{aligned}
B_n(z) &= \frac{1}{h_{n-1}}\int_0^\infty \left[\frac{\alpha}{zx} + \frac{\beta}{(z+1)(x+1)}\right] P_n(x)P_{n-1}(x)w(x)dx \\
&= \frac{\alpha}{zh_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x}dx + \frac{\beta}{(z+1)h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x+1}dx.
\end{aligned}$$

Again, these two above integrals in the previous equality have a simple relation via integration by parts.

Thus,

$$\frac{\beta}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x+1}dx \;=\; \frac{\beta}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)x^\alpha(x+1)^\beta e^{-tx}}{x+1}dx$$

$$= \frac{1}{h_n}\int_0^\infty P_n(x)P_{n-1}(x)x^\alpha e^{-tx}d(x+1)^\beta$$

$$= -\frac{1}{h_n}\int_0^\infty \left(P_n(x)P_{n-1}(x)x^\alpha e^{-tx}\right)'(x+1)^\beta dx$$

$$= \frac{-1}{h_{n-1}}\int_0^\infty P_n'(x)P_{n-1}(x)w(x)dx - \frac{1}{h_{n-1}}\int_0^\infty P_n(x)P_{n-1}'(x)w(x)dx$$

$$- \frac{\alpha}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x}dx + \frac{t}{h_{n-1}}\int_0^\infty P_n(x)P_{n-1}(x)w(x)dx$$

$$= -n - \frac{\alpha}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x}dx,$$

so that

$$\frac{\beta}{h_{n-1}}\int_0^\infty \frac{P_n(x)P_{n-1}(x)w(x)}{x+1}dx = -n - r_n.$$

Then, it follows that

$$B_n(z) = \frac{r_n}{z} - \frac{n+r_n}{z+1},$$

which is the formula (3.17).

This completes the proof. □

In what follows, we are going to substitute (3.16) and (3.17) into the supplementary conditions (S_1), (S_2) and (S_2') in the Theorem 3.2 in order to produce identities which will be important for derivation of our Painlevé equation as we will see it in the next section.

Substituting (3.16) and (3.17) into (S_1), we get

$$\frac{r_{n+1}}{z} - \frac{n+1+r_{n+1}}{z+1} + \frac{r_n}{z} - \frac{n+r_n}{z+1} = (z-b_n)\left(\frac{R_n}{z} + \frac{t-R_n}{z+1}\right) - t + \frac{\alpha}{z} + \frac{\beta}{z+1}.$$

Comparing the coefficients of $\frac{1}{z}$ and $\frac{1}{z+1}$ on both sides, we get the two following equations

$$r_{n+1} + r_n = \alpha - b_n R_n, \tag{3.20}$$

$$-\left(r_{n+1} + r_n\right) = -tb_n + b_n R_n + R_n + 2n + 1 - t + \beta. \tag{3.21}$$

Adding (3.20) and (3.21) gives

$$tb_n = 2n + 1 + \alpha + \beta - t + R_n \tag{3.22}$$

and substracting (3.21) from (3.20), we find

$$r_{n+1} + r_n = -n + \frac{\alpha - \beta + t - 1 - R_n + (t - 2R_n)b_n}{2}. \tag{3.23}$$

Similarly, substituting (3.16) and (3.17) into (S_2), we obtain

$$c_{n+1}\left(\frac{R_{n+1}}{z} + \frac{t - R_{n+1}}{z+1}\right) - c_n\left(\frac{R_{n-1}}{z} + \frac{t - R_{n-1}}{z+1}\right) = 1 + (z - b_n)\left(\frac{r_{n+1}}{z} - \frac{n+1+r_{n+1}}{z+1} - \frac{r_n}{z} + \frac{n + r_n}{z+1}\right).$$

Comparing the coefficients of $\frac{1}{z}$ and $\frac{1}{z+1}$ on both sides, we get

$$c_{n+1}R_{n+1} - c_nR_{n-1} = -b_n(r_{n+1} - r_n),$$

$$c_{n+1}(t - R_{n+1}) - c_n(t - R_{n-1}) = b_n + b_n(r_{n+1} - r_n) + r_{n+1} - r_n + 1.$$

The combination of these above two equations gives

$$t(c_{n+1} - c_n) = b_n + r_{n+1} - r_n + 1.$$

Rewriting the previous above formula yields

$$b_n = t(c_{n+1} - c_n) - (r_{n+1} - r_n) - 1. \tag{3.24}$$

In view of (3.6), a telescopic sum of (3.24) gives

$$tc_n = n + r_n - p(n, t), \tag{3.25}$$

where we have made use of the initial conditions $r_0(t) := 0$.

Substituting (3.16) and (3.17) into (S_2'), we get

$$\left(\frac{r_n}{z}-\frac{n+r_n}{z+1}\right)^2+\left(t-\frac{\alpha}{z}-\frac{\beta}{z+1}\right)+\sum_{j=0}^{n-1}\left(\frac{R_j}{z}+\frac{t-R_j}{z+1}\right)$$

$$=c_n\left(\frac{R_n}{z}+\frac{t-R_n}{z+1}\right)\left(\frac{R_{n-1}}{z}+\frac{t-R_{n-1}}{z+1}\right),$$

which can be rewritten again as

$$\frac{r_n^2-\alpha r_n}{z^2}+\frac{(\alpha-2r_n)(n+r_n)-\beta r_n}{z(z+1)}+\frac{tr_n+\sum_{j=0}^{n-1}R_j}{z}$$
$$+\frac{\sum_{j=0}^{n-1}(t-R_j)-t(n+r_n)}{z+1}+\frac{\beta(n+r_n)+(n+r_n)^2}{(z+1)^2}$$
$$=c_n\left(\frac{R_nR_{n-1}}{z^2}+\frac{R_n(t-R_{n-1})+(t-R_n)R_{n-1}}{z(z+1)}+\frac{(t-R_n)(t-R_{n-1})}{(z+1)^2}\right).$$

Comparing the coefficients of $\frac{1}{z^2}$, $\frac{1}{z(z+1)}$, $\frac{1}{z}$, $\frac{1}{z+1}$ and $\frac{1}{(z+1)^2}$ on both sides respectively, we find

$$r_n^2-\alpha r_n=c_nR_nR_{n-1}, \tag{3.26}$$

$$(\alpha-2r_n)(n+r_n)-\beta r_n=c_nR_n(t-R_{n-1})+c_n(t-R_n)R_{n-1}, \tag{3.27}$$

$$tr_n+\sum_{j=0}^{n-1}R_j=0, \tag{3.28}$$

$$\sum_{j=0}^{n-1}(t-R_j)-t(n+r_n)=0, \tag{3.29}$$

$$(n+r_n)^2+\beta(n+r_n)=c_n(t-R_n)(t-R_{n-1}). \tag{3.30}$$

Remark that the equation (3.29) is equal to the equation (3.28).

Using (3.26) to eliminate R_{n-1} in (3.27) and (3.30), then the above equations become

$$r_n^2-\alpha r_n=c_nR_nR_{n-1}, \tag{3.31}$$

$$tc_n=n\alpha-(2n+\alpha+\beta)r_n-\frac{t}{R_n}(r_n^2-\alpha r_n), \tag{3.32}$$

$$\sum_{j=0}^{n-1} R_j = -tr_n, \tag{3.33}$$

$$t(t-R_n)c_n = n(n+\beta) + (2n+\alpha+\beta)r_n + \frac{t}{R_n}(r_n^2 - \alpha r_n). \tag{3.34}$$

Remark 3.2 Observe also that in the above manipulations, we have arrived at expressing the recurrence coefficients b_n and c_n in terms of the auxiliary quantities R_n and r_n. This will be important for us in the following.

Remark 3.3 Observe that the combination of (3.32) and (3.34) gives

$$t^2 c_n + tc_n(1-R_n) = n(n+\alpha+\beta). \tag{3.35}$$

To check our results, we can return to pure Laguerre by setting $\beta = 0$ and $t = 1$. Then, taking $\beta = 0$ and $t = 1$, the above formula (3.35) becomes

$$c_n(1) + c_n(1)(1 - R_n(1)) = n(n+\alpha).$$

From (3.18), a simple computation by integration by parts of $R_n(1)$ for $\beta = 0$ gives $R_n(1) = 1$.
Hence, it follows that

$$c_n(1) = n(n+\alpha).$$

From (3.22), for $\beta = 0$ and $t = 1$, we also obtain

$$b_n(1) = 2n + 1 + \alpha.$$

These above expressions of $b_n(1)$ and $c_n(1)$ are the recurrence coefficients of monic Laguerre polynomials as we have seen it in the Chapter 1. See the formulas (1.31) and (1.32).

To end this section, we point out that the quantities R_n and r_n in the above equations satisfy coupled nonlinear first-order difference equations which are the discrete Painlevé equations. More precisely we have the following proposition:

Proposition 3.2 *The quantities R_n and r_n satisfy the following discrete Painlevé equations:*

$$t(r_{n+1}+r_n) = -R_n^2 - (2n+1+\alpha+\beta-t)R_n + \alpha t, \tag{3.36}$$

$$n(n+\beta) + (2n+\alpha+\beta)r_n = r_n(r_n-\alpha)\left(\frac{t}{R_n} - \frac{t^2}{R_nR_{n-1}} + \frac{t}{R_{n-1}}\right). \tag{3.37}$$

Proof. From (3.20), we get

$$b_n = \frac{\alpha-(r_{n+1}+r_n)}{R_n}$$

and substituting this in (3.22) the equation (3.36) follows.
From (3.26), we get

$$c_n = \frac{r_n^2-\alpha r_n}{R_nR_{n-1}}$$

and substituting this into (3.34) the equation (3.37) follows and this ends the proof. □

3.5 Coupled Riccati equations and Painlevé V

We start this section from taking a derivative with respect to t in the following equation

$$\int_0^\infty P_n^2(x,t)x^\alpha(x+1)^\beta e^{-tx}dx = h_n(t),$$

which produces

$$h_n'(t) = 2\underbrace{\int_0^\infty P_n(x,t)P_n'(x,t)w(x,t)dx}_{0} - \int_0^\infty xP_n^2(x,t)w(x,t)dx.$$

From (3.2), we find

$$\int_0^\infty xP_n^2(x,t)w(x,t)dx = b_nh_n(t).$$

Hence, it follows that

$$h_n'(t) = -b_nh_n(t),$$

or

$$\frac{h_n'(t)}{h_n(t)} = -b_n, \tag{3.38}$$

that is

$$\frac{d}{dt} \ln h_n(t) = -b_n. \tag{3.39}$$

Lemma 3.4. *The recurrence coefficients b_n and c_n satisfy the following differential equations*

$$\frac{dc_n}{dt} = c_n(b_{n-1} - b_n), \tag{3.40}$$

$$\frac{db_n}{dt} = c_n - c_{n+1}. \tag{3.41}$$

Proof. From (3.5), we get

$$\begin{aligned} \frac{dc_n}{dt} &= \frac{h_n'(t)h_{n-1}(t) - h_n(t)h_{n-1}'(t)}{h_{n-1}^2(t)} \\ &= \frac{h_n'(t)}{h_n(t)} \frac{h_n(t)}{h_{n-1}(t)} - \frac{h_n(t)}{h_{n-1}(t)} \frac{h_{n-1}'(t)}{h_{n-1}(t)}. \end{aligned}$$

Using (3.38) and (3.5), it follows that

$$\frac{dc_n}{dt} = -b_n c_n + c_n b_{n-1},$$

so that

$$\frac{dc_n}{dt} = c_n(b_{n-1} - b_n),$$

which is (3.40).

To prove (3.41), we begin with taking a derivative with respect to t in the following equation

$$\int_0^\infty P_n(x,t)P_{n-1}(x,t)x^\alpha(x+1)^\beta e^{-tx}dx = 0,$$

which produces

$$\underbrace{\int_0^\infty \frac{d}{dt}P_n(x,t)P_{n-1}(x,t)w(x,t)dx}_{\frac{d}{dt}p(n,t)h_{n-1}} + \underbrace{\int_0^\infty P_n(x,t)\frac{d}{dt}P_{n-1}(x,t)w(x,t)dx}_{0}$$
$$- \int_0^\infty xP_n(x,t)P_{n-1}(x,t)w(x,t)dx = 0,$$

and from (3.2) with the orthogonality condition (3.1), we find

$$\int_0^\infty xP_n(x,t)P_{n-1}(x,t)w(x,t)dx = c_n h_{n-1}.$$

Plugging this into the above equation gives

$$\frac{d}{dt}p(n,t)h_{n-1} - c_n h_{n-1} = 0.$$

Then,

$$\frac{dp(n,t)}{dt} = c_n. \tag{3.42}$$

In view of (3.3), this equation (3.42) yields

$$\frac{db_n}{dt} = c_n - c_{n+1},$$

which is (3.41). □

Remark 3.4 The equations (3.40) and (3.41) are the Toda equations.

Now with the aid of the previous lemma, we obtain the coupled Riccati equations satisfied by the auxiliary quantities R_n and r_n. More precisely, we have the following lemma.

Lemma 3.5. *The auxiliary quantities R_n and r_n satisfy the following coupled Riccati equations:*

$$t\frac{dR_n}{dt} = -\alpha t + (2n+1+\alpha+\beta-t)R_n + R_n^2 + 2tr_n \tag{3.43}$$

$$\frac{dr_n}{dt} = \frac{r_n^2 - \alpha r_n}{R_n} - \frac{R_n}{t(t-R_n)}\left[n(n+\beta) + (2n+\alpha+\beta)r_n + \frac{t}{R_n}(r_n^2 - \alpha r_n)\right], \tag{3.44}$$

where R_n and r_n are defined in (3.18) and (3.19), respectively.

Proof. In the view of (3.41), the equation(2.9) gives

$$b_n = -t\frac{db_n}{dt} - (r_{n+1} - r_n) - 1.$$

Solving (3.23) for r_{n+1} and plugging it into the above equation, we get

$$t\frac{db_n}{dt} + b_n = n - 1 + 2r_n - \frac{\alpha - \beta + t - 1 - R_n + (t - 2R_n)b_n}{2}.$$

From (3.22), expressing b_n in terms of R_n, we obtain

$$b_n = \frac{2n + 1 + \alpha + \beta}{t} - 1 + \frac{R_n}{t}$$

and substituting this into the previous equation, we get

$$t\left[-\frac{(2n + 1 + \alpha + \beta)}{t^2} + \frac{t\frac{dR_n}{dt} - R_n}{t^2}\right] + \frac{2n + 1 + \alpha + \beta}{t} - 1 + \frac{R_n}{t} = n - 1 + 2r_n$$

$$- \frac{\alpha - \beta + t - 1 - R_n}{2} - \frac{1}{2}(t - 2R_n)\left(\frac{2n + 1 + \alpha + \beta}{t} - 1 + \frac{R_n}{t}\right).$$

After some simple simplifications, we find

$$t\frac{dR_n}{dt} = -\alpha t + (2n + 1 + \alpha + \beta - t)R_n + R_n^2 + 2tr_n$$

which is (3.43).

Next, we give the proof of (3.44).
For this purpose, recall that in (3.6), (3.42) and (3.7) we have, respectively

$$-\sum_{j=0}^{n-1} b_j = p(n, t),$$

$$p'(n, t) = c_n$$

and

$$D_n = \prod_{j=0}^{n-1} h_j.$$

It follows that the logarithmic derivative of the Hankel determinant yields

$$\begin{aligned}\frac{d}{dt}\ln D_n &= \frac{d}{dt}\left(\ln\prod_{j=0}^{n-1}h_j\right)\\ &= \frac{d}{dt}\left(\sum_{j=0}^{n-1}\ln h_j\right)\\ &= \sum_{j=0}^{n-1}\frac{d}{dt}\ln h_j\end{aligned}$$

that is

$$\frac{d}{dt}\ln D_n = -\sum_{j=0}^{n-1}b_j,$$

where we have used (3.39) in this last formula.
Therefore,

$$\frac{d}{dt}\ln D_n = -\sum_{j=0}^{n-1}b_j = p(n,t).$$

Hence, in view of (3.25), we obtain

$$\frac{d}{dt}\ln D_n = n + r_n - tc_n = p(n,t).$$

By taking a derivative with respect to t in the previous equation, we get

$$\frac{dr_n}{dt} - c_n - t\frac{dc_n}{dt} = \frac{dp(n,t)}{dt},$$

so that

$$\frac{dr_n}{dt} = 2c_n + tc_n(b_{n-1} - b_n), \tag{3.45}$$

where we have used the first Toda equation (3.40) of the lemma 3.4 to replace $\frac{dc_n}{dt}$.

A simple computation with (3.22) gives

$$t(b_{n-1} - b_n) = -2 + R_{n-1} - R_n.$$

Hence, using this, the equation (3.45) becomes

$$\begin{aligned}\frac{dr_n}{dt} &= 2c_n + c_n(-2 + R_{n-1} - R_n)\\ &= c_n(R_{n-1} - R_n)\end{aligned}$$

and substituting $c_n R_{n-1}$ with the aid of (3.31), we obtain

$$\frac{dr_n}{dt} = \frac{r_n^2 - \alpha r_n}{R_n} - c_n R_n.$$

Finally, we replace c_n by using (3.34) and we find

$$\frac{dr_n}{dt} = \frac{r_n^2 - \alpha r_n}{R_n} - \frac{R_n}{t(t-R_n)}\left[n(n+\beta) + (2n+\alpha+\beta)r_n + \frac{t}{R_n}(r_n^2 - \alpha r_n)\right]$$

which is (3.44). □

In what follows, to arrive at our main result, we will devote our efforts to deriving the second order ODE for which we make a certain linear fractional transformation to obtain the Painlevé V. For this purpose, our idea is to eliminate r_n in the previous lemma. More precisely, we present our main result in the following theorem.

Theorem 3.6. *The quantity R_n satisfies the following non-linear seconder order differential equation,*

$$R_n'' = \left(\frac{1}{R_n} - \frac{1}{t-R_n}\right)\frac{(R_n')^2}{2} + \left(-1 + \frac{t}{t-R_n}\right)\frac{R_n'}{t} + (R_n + 2n + 1 + \alpha + \beta)\frac{R_n^2}{t^2}$$

$$+\left[\frac{-3R_n}{2} - (2n+1+\alpha+\beta)\right]\frac{R_n}{t} + \frac{R_n}{2} + \frac{\beta^2 - 1}{2(t-R_n)} - \frac{\alpha^2}{2R_n} + \frac{\alpha^2 - \beta^2 + 1}{2t}. \quad (3.46)$$

Let $y(t) := 1 - \frac{t}{R_n(t)}$, then $y(t)$ satisfies a second order differential equation

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2}{2}y - \frac{\beta^2/2}{y}\right) + (2n+1+\alpha+\beta)\frac{y}{t} - \frac{y(y+1)}{2(y-1)}, \quad (3.47)$$

which is a particular Painlevé V, i.e., $P_V(\frac{\alpha^2}{2}, -\frac{\beta^2}{2}, 2n+1+\alpha+\beta, -\frac{1}{2})$, following the notation in (2.6).

Proof. We begin with taking the derivative with respect to t in (3.43) and we next use (3.44) to obtain

$$tR_n'' = (2n+1+\alpha+\beta-t+2R_n)R_n' - \alpha - R_n - \frac{2n(n+\beta)}{t-R_n}R_n$$

$$+\,2\left[1-\frac{(2n+\alpha+\beta)}{t-R_n}R_n - \alpha t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]r_n + 2t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)r_n^2.$$

Let

$$(*) := 2\left[1-\frac{(2n+\alpha+\beta)}{t-R_n}R_n - \alpha t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]r_n \tag{3.48}$$

and

$$(**) := 2t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)r_n^2. \tag{3.49}$$

Hence, the above equation becomes

$$tR_n'' = (2n+1+\alpha+\beta-t+2R_n)R_n' + \left[-1-\frac{2n(n+\beta)}{t-R_n}\right]R_n - \alpha + (*) + (**). \tag{3.50}$$

Solving (3.43) for r_n, we obtain

$$r_n = \frac{R_n'}{2} + \frac{\alpha}{2} - \frac{1}{2}(2n+1+\alpha+\beta)\frac{R_n}{t} + \frac{R_n}{2} - \frac{R_n^2}{2t}. \tag{3.51}$$

Inserting this equation (3.51) in (3.48), we have

$$(*) = 2\left[1-\frac{(2n+\alpha+\beta)R_n}{t-R_n} - \alpha t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]$$

$$\times\left[\frac{R_n'}{2} + \frac{\alpha}{2} - \frac{1}{2}(2n+1+\alpha+\beta)\frac{R_n}{t} + \frac{R_n}{2} - \frac{R_n^2}{2t}\right].$$

After some simplifications, we obtain

$$\begin{aligned}(*) \;=\; & \left[1-\frac{(2n+\alpha+\beta)}{t-R_n}R_n - \alpha t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]R_n' + \frac{(2n+\alpha+\beta)}{t-R_n}\frac{R_n^3}{t} \\ & + \left[-1+\frac{(2n+\alpha+\beta)(2n+1+\alpha+\beta)}{t-R_n}\right]\frac{R_n^2}{t} - (2n+1+\alpha+\beta)\frac{R_n}{t} \\ & + \left[\frac{-(2n+\alpha+\beta)}{t-R_n} + \alpha\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]R_n^2 \\ & + \left[1-\frac{\alpha(2n+\alpha+\beta)}{t-R_n} + (\alpha(2n+1+\alpha+\beta)-\alpha t)\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]R_n \\ & -\alpha - \alpha^2 t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\end{aligned} \tag{3.52}$$

and inserting (3.51) in (3.49), we have

$$(**) = 2t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\left[\frac{R_n'}{2}+\frac{\alpha}{2}-\frac{1}{2}(2n+1+\alpha+\beta)\frac{R_n}{t}+\frac{R_n}{2}-\frac{R_n^2}{2t}\right]^2,$$

which after some straightforward simplifications gives again

$$(**) = t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{(R_n')^2}{2}$$
$$+[\alpha+(\tfrac{1}{R_n}-\tfrac{1}{t-R_n})-(2n+1+\alpha+\beta)(\tfrac{1}{R_n}-\tfrac{1}{t-R_n})R_n+t(\tfrac{1}{R_n}-\tfrac{1}{t-R_n})R_n-(\tfrac{1}{R_n}-\tfrac{1}{t-R_n})R_n^2]R_n'$$
$$+\frac{1}{2}\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{R_n^4}{t}+(2n+1+\alpha+\beta)\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{R_n^3}{t}$$
$$+\frac{1}{2}(2n+1+\alpha+\beta)^2\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{R_n^2}{t}-\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n^3$$
$$+[\frac{t}{2}-(2n+1+2\alpha+\beta)]\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n^2+[\alpha t-\alpha(2n+1+\alpha+\beta)]\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n$$
$$+\frac{\alpha^2 t}{2}\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right). \tag{3.53}$$

Substituting (3.52) and (3.53) into (3.50), we find

$$tR_n'' =$$
$$t\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{(R_n')^2}{2}+[2n+1+\alpha+\beta+2R_n-\frac{(2n+\alpha+\beta)R_n}{t-R_n}$$
$$-(2n+1+\alpha+\beta)\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n-\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n^2]R_n'$$
$$+\frac{1}{2}\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\frac{R_n^4}{t}+\left[\frac{(2n+\alpha+\beta)}{t}+(2n+1+\alpha+\beta)\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]\frac{R_n^3}{t}$$
$$+\left[-1+\frac{(2n+\alpha+\beta)(2n+1+\alpha+\beta)}{t-R_n}+\frac{1}{2}(2n+1+\alpha+\beta)^2\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]\frac{R_n^2}{t}$$
$$-(2n+1+\alpha+\beta)\frac{R_n}{t}-\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)R_n^3$$
$$+\left[\frac{-(2n+\alpha+\beta)}{t-R_n}+\alpha(\frac{1}{R_n}-\frac{1}{t-R_n})+(\frac{t}{2}-(2n+1+\alpha+\beta))\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right)\right]R_n^2$$
$$+\left[\frac{-2n(n+\beta)}{t-R_n}-\frac{\alpha(2n+\alpha+\beta)}{t-R_n}\right]R_n-\frac{\alpha^2 t}{2}\left(\frac{1}{R_n}-\frac{1}{t-R_n}\right).$$

A little bit simplification of this equation yields

$$R_n'' = \left(\frac{1}{R_n} - \frac{1}{t-R_n}\right)\frac{(R_n')^2}{2} + \left(-1 + \frac{t}{t-R_n}\right)\frac{R_n'}{t} + (R_n + 2n + 1 + \alpha + \beta)\frac{R_n^2}{t^2}$$

$$+ \left[\frac{-3R_n}{2} - (2n+1+\alpha+\beta)\right]\frac{R_n}{t} + \frac{R_n}{2} + \frac{\beta^2-1}{2(t-R_n)} - \frac{\alpha^2}{2R_n} + \frac{\alpha^2-\beta^2+1}{2t}.$$

which is the desired equation (3.46).

Now to end, we prove (3.47).

Let

$$y(t) := 1 - \frac{t}{R_n(t)},$$

then,

$$R_n' = \frac{ty'}{(y-1)^2} - \frac{1}{y-1},$$

$$R_n'' = \frac{ty''}{(y-1)^2} + \frac{2ty'}{(y-1)^2} - \frac{2t(y')^2}{(y-1)^3}$$

and

$$(R_n')^2 = \frac{t^2(y')^2}{(y-1)^4} - \frac{2ty'}{(y-1)^3} + \frac{1}{(y-1)^2}.$$

Substituting these above expressions into (3.46), we have

$$\frac{ty''}{(y-1)^2} = \frac{t(3y-1)}{2y(y-1)^3}(y')^2 - \frac{y'}{(y-1)^2} - \frac{1}{2t(y-1)} - \frac{1}{2ty(y-1)} + \frac{1}{t(y-1)} - \frac{1}{ty} - \frac{t}{(y-1)^3}$$

$$+\frac{(2n+1+\alpha+\beta)}{(y-1)^2} - \frac{3t}{2(y-1)^2} + \frac{(2n+1+\alpha+\beta)}{y-1} - \frac{t}{2(y-1)}$$
$$+\frac{(\beta^2-1)(y-1)}{2y} + \frac{\alpha^2(y-1)}{2t} + \frac{\alpha^2-\beta^2+1}{2t}.$$

Multiplying the previous equation by $(y-1)^2$ and dividing by t on both sides, we obtain

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y}{t}$$

$$+\frac{(y-1)^2}{t^2}\left[\frac{(\beta^2-1)(y-1)}{2y} - \frac{1}{2y(y-1)} - \frac{1}{y} - \frac{1}{2(y-1)} + \frac{1}{y-1} + \frac{\alpha^2(y-1)}{2} + \frac{\alpha^2-\beta^2+1}{2}\right]$$

$$+(2n+1+\alpha+\beta)\frac{y}{t} - \frac{1}{y-1} - \frac{3}{2} - \frac{(y-1)}{2},$$

so that

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2}{2}y - \frac{\beta^2/2}{y}\right) + (2n+1+\alpha+\beta)\frac{y}{t} - \frac{y(y+1)}{2(y-1)},$$

which is the Painlevé equation V in (3.47). This finishes our proof. □

Conclusion and outlooks

This thesis is concerned in the link between orthogonal polynomials and Painlevé equations. More precisely, we recall that our main goal was to find a Painlevé equation from a given weight function.

The Chapter 1 deals with the orthogonal polynomials on the real line and some of their properties in particular those of Laguerre polynomials.

In Chapter 2, we have introduced the Painlevé equations and some of their mathematical properties are discussed in particular those of fifth equation.

In Chapter 3 which concerns our main contribution, we investigated the Hankel determinant generated by the deformed Laguerre weight function for orthogonal polynomials and we derived a Painlevé V as it was shown in Theorem 3.6.

Although we were able to find the Painlevé equation from our deformed Laguerre weight, we remark that there are several ways to deform the Laguerre weight function but all deformed Laguerre weights do not lead to the Painlevé equation. Therefore, further reasearch is needed . Thus, it would be of a great importance to find others new deformed Laguerre weight functions which lead to a Painlevé equation and also a necessary and sufficient condition for an arbitrary weight function to lead to a Painlevé equation.

Appendix

In this appendix we refer to [53]

A.1 $q-$discrete Painlevé equations

$$q\text{-}P_{II} \quad (x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\lambda_n \lambda_{n-1} x_n}{\alpha^2(x_n - \alpha\lambda_n)},$$

$$q\text{-}P_{II} \quad x_{n+1}x_{n-1} = \alpha\lambda_n \frac{\lambda_n + x_n}{x_n(x_n - 1)},$$

$$q\text{-}P_{III} \quad x_{n+1}x_{n-1} = \frac{(x_n + \alpha)(x_n + \beta)}{(\gamma\lambda_n x_n + 1)(\delta\lambda_n x_n + 1)},$$

$$q\text{-}P_{IV} \quad (x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\gamma\delta(x_n + \alpha)(x_n + 1/\alpha)(x_n + \beta)(x_n + 1/\beta)}{(\gamma\lambda x_n + 1)(\delta\lambda x_n + 1)},$$

$$q\text{-}P_{V} \quad (x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\gamma\delta\lambda_n^2(x_n - \alpha)(x_n - 1/\alpha)(x_n - \beta)(x_n - 1/\beta)}{(x_n - \gamma\lambda_n)(x_n - \delta\lambda_n)},$$

$$q\text{-}P_{VI} \quad \frac{(x_n x_{n+1} - \lambda_n\lambda_{n+1})(x_n x_{n-1} - \lambda_n\lambda_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} =$$
$$\frac{(x_n - \alpha\lambda_n)(x_n - \lambda_n/\alpha)(x_n - \beta\lambda_n)(x_n - \lambda_n/\beta)}{(x_n - \gamma)(x_n - 1/\gamma)(x_n - \delta)(x_n - 1/\delta)},$$

with $a + b + c + d = 0$, $p + q + r + s = 0$,
where $\lambda_n = \lambda_0 q^n$ and α, β, γ and δ are constants.

A.2 Asymmetric discrete Painlevé equations

$$x_{n+1} = \frac{f_1(y_n) + x_n f_2(y_n)}{f_3(y_n) + x_n f_4(y_n)}, \qquad y_{n-1} = \frac{g_1(x_n) + y_n g_2(x_n)}{g_3(x_n) + y_n g_4(x_n)}.$$

α-d-P_I $\quad x_{n+1} + x_n + y_n = \delta + \dfrac{z_n - \gamma}{y_n},$

$$y_{n+1} + y_{n-1} + x_n = \delta + \frac{z_{n+1/2} + \gamma}{x_n},$$

α-d-P_{II} $\quad x_{n+1} + x_n = \dfrac{2(y_n z_n + \gamma)}{1 - y_n^2},$

$$y_{n+1} + y_n = \frac{2(y_n z_{n+1/2} - \delta)}{1 - x_n^2},$$

α-d-P_{III} $\quad x_{n+1} x_n = \dfrac{(y_n - q^n a)(y_n - q^n b)}{(y_n - c)(y_n - d)},$

$$y_n y_{n-1} = \frac{(x_n - q^n \alpha)(x_n - q^n \beta)}{(x_n - \gamma)(x_n - \delta)}, \qquad \left(\frac{\alpha\beta}{\gamma\delta} = q\frac{ab}{cd}\right),$$

α-d-P_{IV} $\quad (y_n + x_n)(x_{n+1} + y_n) = \dfrac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(y_n + \gamma - z_n)(y_n - \gamma - z_n)},$

$$(y_n + x_n)(x_n + y_{n-1}) = \frac{(x_n + a)(x_n + b)(x_n + c)(x_n + d)}{(x_n + \delta - z_{n+1/2})(x_n - \delta - z_{n+1/2})},$$

with $a + b + c + d = 0$.

$$\alpha\text{-}d\text{-}P_V \quad \frac{(y_n + x_{n+1} - z_n - z_{n+1})(x_{n+1} + y_n - z_n - z_{n-1})}{(y_n + x_n)(x_{n+1} + y_n)}$$

$$= \frac{(y_n - z_n - a)(y_n - z_n - b)(y_n - z_n - c)(y_n - z_n - d)}{(y_n - p)(y_n - q)(y_n - r)(y_n - s)},$$

$$\frac{(y_n + x_{n+1} - z_n - z_{n-1/2})(x_{n+1} + y_n - z_{n-1/2} - z_{n-1})}{(y_n + x_n)(x_n + y_{n-1})}$$

$$= \frac{(x_n - z_{n-1/2} + a)(x_n - z_{n-1/2} + b)(x_n - z_{n-1/2} + c)(x_n - z_{n-1/2} + d)}{(x_n + p)(x_n + q)(x_n + r)(x_n + s)},$$

$$\alpha\text{-}d\text{-}P_V \quad (x_n y_n - 1)(x_{n-1} y_n - 1) = q^{2n} \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(q^n - \mathcal{K} y_n)(q^n - y_n/\mathcal{K})},$$

$$(x_n y_n - 1)(x_n y_{n+1} - 1) = q^{2n+1} \frac{(x_n - 1/a)(x_n - 1/b)(x_n - 1/c)(x_n - 1/d)}{(q^{n+1/2} - \mu y_n)(q^{n+1/2} - y_n/\mu)},$$

with $abcd = 0$.

$$\alpha\text{-}d\text{-}P_{VI} \quad \begin{cases} x_n x_{n+1} = \dfrac{\beta_3 \beta_4 (y_n - q^n \alpha_1)(y_n - q^n \alpha_2)}{(y_n - \alpha_3)(y_n - \alpha_4)}, \\ y_n y_{n-1} = \dfrac{\alpha_3 \alpha_4 (x_n - q^n \beta_1)(x_n - q^n \beta_2)}{(x_n - \beta_3)(x_n - \beta_4)}, \end{cases}$$

with $\dfrac{\alpha_1 \alpha_2}{\alpha_3 \alpha_4} = q \dfrac{\beta_1 \beta_2}{\beta_3 \beta_4}$,
where $z_n = \alpha n + \beta$ and $a, b, c, d, p, q, r, s, \alpha, \beta, \gamma$ and δ are constants.

A.3 Alternative Painlevé equations

a-d-P$_I$ $x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \mu,$

$$\frac{z_n}{x_{n+1} + x_{n-1}} + \frac{z_{n-1}}{x_n + x_{n-1}} = -x_n^2 + \gamma,$$

$$x_{n+1} + x_{n-1} = \frac{z_n}{x_n} + \frac{\gamma}{x_n^2},$$

$$x_{n+1} + x_{n-1} = \frac{z_n}{x_n} + \gamma,$$

$$x_{n+1}x_{n-1} = \frac{\exp(z_n)}{x_n} + \frac{\gamma}{x_n^2},$$

a-d-P$_{II}$ $x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2},$

$$\frac{z_n}{x_{n+1}x_n + 1} + \frac{z_{n-1}}{x_n x_{n-1} + 1} = -x_n + \frac{1}{x_n} + z_n + \gamma,$$

a-d-P$_V$ $\frac{(x_{n+1} + x_n - 2z_{n+1})(x_n + x_{n-1} - 2z_n)}{(x_{n+1} + x_n)(x_n + x_{n-1})} = \frac{(x_n - z_{n+1/2})^2 - \mathcal{K}^2}{(x_n - \gamma)^2},$

$$\frac{z_{n+1/2}}{1 - x_n x_{n+1}} + \frac{z_{n-1/2}}{1 - x_n x_{n-1}} = \mu + z_n + \frac{\mathcal{K} x_n}{(1 + x_n)^2} + \frac{1 - x_n}{1 + x_n}\left[\frac{1}{2} z_n + (-1)^n \gamma\right],$$

where $z_n = \alpha n + \beta$ and α, β, γ and δ are constants.

A.4 Other discrete Painlevé equations

$$d\text{-}P_I \quad x_{n+1}+x_n=\frac{y_nz_n+\gamma}{y_n^2},\quad y_n+y_{n-1}=\frac{x_nz_{n+1/2}+\delta}{x_n^2},$$

$$d\text{-}P_{II} \quad x_{n+1}+x_n=\frac{y_nz_n+\gamma}{y_n^2-\mu^2},\quad y_n+y_{n-1}=\frac{x_nz_{n+1/2}+\delta}{x_n^2-\ mu^2},$$

$$d\text{-}P_{IV} \quad x_nx_{n-1}=\frac{(y_n+z_n-a)(y_n+z_n-b)}{y_n^2-\gamma^2},\quad y_n+y_{n+1}=\frac{z_{n+1/2}+c}{x_n\gamma+1}-\frac{z_{n+1/2}+d}{x_n/\gamma+1},$$

with $a+b+c+d=0$.

$$d\text{-}P_{IV} \quad x_nx_{n-1}=\frac{a(y_n+z_n-b)}{y_n^2-\gamma^2},\quad y_n+y_{n+1}=\frac{c}{x_n}+\frac{z_{n+1/2}+d}{x_n-1},$$

$$x_n+x_{n-1}=\frac{z_n+\mu}{1+y_n/t}+\frac{z_n-\mu}{1+ty_n},\quad y_ny_{n+1}=\frac{(x_n-z_{n+1/2})^2-\mathcal{K}^2}{x_n^2-\gamma^2},$$

$$x_n+x_{n-1}=\frac{\gamma}{1+y_n}+\frac{z_n+\delta}{1-y_n},\quad y_ny_{n+1}=\gamma\frac{x_n-z_n}{x_n^2-\mu^2},$$

where $z_n=\alpha n+\beta$ and $a,b,c,d,p,q,r,s,\alpha,\beta,\gamma,\delta,\delta,\mathcal{K}$ and μ are constants.

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