2024-06

Equations d’évolution hyperbolique sur les espaces berwaldiens

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UB, ENS

ECOLE NORMALE SUPERIEURE ET
UNIVERSITE DU BURUNDI

MASTER CONJOINT EN DIDACTIQUE DES SCIENCES
OPTION: MATHEMATIQUES

EQUATIONS D'EVOLUTION HYPERBOLIQUE SUR LES ESPACES BERWALDIENS

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Mémoire Présenté et soutenu publiquement en vue de l' obtention du grade de **Master en Didactique des Sciences Mathématiques.**

Bujumbura, Juin 2024

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Dédicaces

A mon père pour sa rigueur paternelle ;
A ma regrettée mère qui n'a pas savouré le fruit de son fils ;
A mes sœurs pour leur soutien sans faille ;
A tous mes oncles et tantes ;
A mes cousins et cousines ;
A mes camarades de promotion.

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Remerciements

Au terme de ce mémoire, j'éprouve un agréable devoir d'exprimer mes vifs remerciements de reconnaissances à toute personne qui, de près ou de loin a contribué à la réalisation de ce mémoire.

Mes vifs et sincères remerciements vont d'abord à l'endroit du tout Puissant pour m'avoir guidé et assisté durant mon parcours d'études.
Ma reconnaissance s'adresse plus particulièrement à mon encadreur Pr. Gilbert NIBARUTA, qui, malgré ses nombreuses activités, a accepté de diriger ce mémoire. Il m'a fait découvrir le sens de la recherche en Mathématiques. Ses précieux conseils, sa disponibilité permanente, surtout sa rigueur au travail et ses remarques m'ont été sans reproche dans la rédaction de ce mémoire.

De plus, j'adresse mes sentiments de gratitude à tous les enseignants du Département des Sciences Naturelles à l'Ecole Normale Supérieure du Burundi ainsi qu'à ceux de l'Institut de Pédagogie Appliquée (IPA), pour leur meilleure formation.

Je tiens également à remercier les membres du jury pour leur présence et pour leur lecture attentive de mon mémoire.

Je ne pourrais jamais oublier le soutien et l'aide des personnes chères de ma famille, mes sœurs et mes amis.

Enfin, que toute personne qui s'est investie d'une façon ou d'une autre à la réussite de ce mémoire accepte mes remerciements.

Résumé

Soit (M, F_t) un espace berwaldien. Dans ce mémoire, les équations d'évolution hyperbolique de la courbure de Ricci et de la courbure scalaire de (M, F_t) sont données localement. De plus, le long du flux géométrique hyperbolique, ces équations sur les espaces berwaldiens, sont des dérivés.

Mots-clés : Espace de Berwald, flux géométrique hyperbolique, équations d'évolution de courbure.

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Abstract

Let (M, F_t) be a berwaldian manifold. In this memory, the hyperbolic evolution equations of Ricci curvature and scalar curvature of (M, F_t) are given locally. Furthermore, along the hyperbolic geometric flow, those equations on berwaldian spaces, are derivatives.

Keywords : berwaldian manifold, hyperbolic geometric flow, curvature evolution equations.

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Avant-propos

Ce mémoire rentre dans le cadre de l'obtention du diplôme de **Master en Didactique des Sciences Mathématiques** et s'oriente dans le domaine de la Géométrie de Finsler.
Il étudiera les équations d'évolution hyperbolique sur les espaces berwaldiens et l'idée de ce mémoire de recherche est venue du constat que les équations d'évolution sur les espaces berwaldiens ont été établies en 2020 sous le flux de Ricci.
En effet, à base des notions sur les espaces berwaldiens et les courbures berwaldiennes, on a pu établir les équations d'évolution hyperbolique des fonctions courbures berwaldiennes le long du flux géométrique.
Cette recherche se veut être une contribution devant permettre de bien comprendre les propriétés ondulatoires des courbures berwaldiennes.

Introduction générale

Les équations d'évolution hyperbolique sont d'intérêt significatif dans de nombreux domaines de Mathématiques et de la Physique. Ces équations permettent de modéliser des phénomènes physiques complexes tenant compte des propriétés géométriques de l'espace dans lequel ils se produisent.

Un flux géométrique est une évolution d'une structure géométrique sous une équation différentielle liée à une fonctionnelle sur un espace, généralement associée à une certaine courbure.

Le sujet du flux de Ricci Hamilton ([10] : $\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)}$ où $g(t)$ est une métrique riemannienne dépendant d'un paramètre t) réside, le plus généralement, dans l'analyse des champs des flux géométriques, qui à son tour se situe dans le domaine de l'analyse des champs des flux géométriques. Dans le flux de Ricci, nous voyons l'unicité de la géométrie et de l'analyse. Entant que le système entièrement non linéaire d'équations aux dérivées partielles paraboliques du second ordre [8], le flux de Ricci semble à bien des égards être une équation très naturelle. De même, puisque l'équation ou le système hyperbolique est l'un des modèles les plus naturels dans la nature nous ressentons que le flux géométrique hyperbolique $\frac{\partial^2}{\partial t^2}g(t) = -2Ric_{g(t)}$, introduit par KONG et Liu [11] en 2007, est également un outil très naturel et utile pour comprendre le caractère ondulatoire des métriques, le phénomène ondulatoire des courbures, l'évolution des variétés et leur structure (voir [11], [6],[13]).

Dans ce travail, l'objectif est d'établir les équations d'évolution hyperbolique des fonctions courbures d'un espace berwaldien sous le flux géométrique hyperbolique. Plus précisement, nous nous sommes concentrés sur l'obtention des formules locales des évolutions en utilisant les techniques de S. Brendle ([3], [4]), voire [15].

Nous supposons que $\{M, g(t), t \in (o, T)\}$ est une famille de variétés berwaldiennes complètes évoluant sous le flux géométrique hyperbolique $\frac{\partial^2}{\partial t^2}g(t) = -2Ric_{F(t)}$. De plus **Ric** et **Scal** désignent la courbure de Ricci et Scalaire de $(M, g(t))$ respectivement.

Le reste de ce mémoire est repartie comme suit :

Dans le premier chapitre, nous donnons quelques notions de bases sur les espaces berwaldiens. Le second chapitre est consacré à l'étude des courbures berwaldiennes très utile dans le troisième chapitre. Dans le troisième chapitre, nous dérivons les équations d'évolution des courbures berwaldiennes le long du flux géométrique hyperbolique.

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Chapitre 1

Notions de base sur les espaces berwaldiens

Soit M une variété différentiable de classe C^{∞} et de dimension n, on désigne par :

1. x : un point de M
2. $\{x^i\}_{i=1,\ldots,n}$: les coordonnées locales de x
3. U : un ouvert de M
4. TM : le fibré tangent de M
5. T_xM : l'espace tangent en x à M
6. (x, y) : un point de TM
7. (x^i, y^i) : les coordonnés locales dans TM.
8. $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\ldots,n}$: sections de base induite par $\{x^i\}$
9. $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}_{i=1,\ldots,n}$: section de base pour TM
10. $\pi : TM \to M; (x, y) \longmapsto x$: une projection naturelle
11. $\pi^{-1}(U)$: un ouvert de TM
12. $TM\backslash\{0\}$: Fibré tangent privée de section nulle.

1.1 Définitions et propriétés

1.1.1 Définition d'une métrique finslérienne

Définition 1.1. *Une métrique finslérienne sur une variété différentiable M est une application $F : TM \to \mathbb{R}^+$ satisfaisant aux trois propriétés suivantes :*

1. *Régularité : $F \in C^{\infty}$.*
2. *Homogénéité : Pour tout $\lambda \in \mathbb{R}^+{}_0, F(x, \lambda y)$=$\lambda F(x, y)$.*
3. *Connexité forte : En chaque point $(x, y) \in TM\backslash 0$, la hessienne $(g_{ij}(x, y))$ de $\frac{1}{2}F^2(x, y)$ dont les éléments*

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{1.1}$$

est définie positive.

Remarque 1.1. *1. Si $y = 0$, alors $F(x, y) = 0$.*

2. *Si $y \neq 0$, alors $F(x, y) \neq 0$.*
3. *Il arrive des fois où $F(x, \lambda y) = \mid \lambda \mid F(x, y)\ \forall \lambda \in \mathbb{R}$, dans ce cas on dit que F est réversible. Dans la suite, nous supposons que F est non réversible.*
4. *La hessienne $(g_{ij}(x, y))$ dont les composantes sont définies en (1.1) détermine un tenseur tenseur-métrique riemannien dépendant de x et/ou de y.*

1.1.2 Propriétés

Lemme 1.1 (Euler). *[1]Soient V un espace vectoriel réel et $f : V \to \mathbb{R}$ une application C^∞ sur $V \setminus \{0\}$. Alors les conditions suivantes sont équivalentes :*

(i) $f(cy) = c^r f(y)$, *f est une fonction homogène de degré r*

(ii) $y^i \frac{\partial f(y)}{\partial y^i} = r f(y)$.

1.1.3 Quelques particularités d'une métrique finslérienne

1. **Métrique riemannienne :** Une métrique riemannienne g sur une variété M est une famille de produits scalaires $\{g_x\}_{x \in M}$ sur chaque espace tangent T_xM : on a $g = (g_{ij}(x)) = \left(g_x\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right)\right)$. Ainsi le tenseur finslérien $(g_{ij}(x, y))$ sera riemannien si la relation suivante est satisfaite

$$g_{ij}(x, y) = g_{ij}(x), \forall x \in M \ \ et \ \ \forall y \in T_xM. \tag{1.2}$$

2. **Métrique minkowskienne :** Une métrique finslérienne F sur M est dite localement Minkowskienne s'il existe un atlas sur M dans lequel le tenseur g est tel que :

$$g_{ij}(x, y) = g_{ij}(y). \tag{1.3}$$

   De plus, F sera minkowskienne si M est un espace vectoriel.

3. **Métriques berwaldiennes :** Le tenseur de composantes données en (1.1), induite par F, est en réalité une métrique riemannienne dépendant d'un point x et d'une direction y. Par conséquent, les symboles de Christoffel relatifs à cette métrique, $g(x, y) = (g_{ij}(x, y))$, dépendent de x et/ou de y. Désignons ces symboles par $\Gamma^k_{ij}(x, y)$.
   Ainsi, une métrique berwaldienne est une métrique finslérienne satisfaisant à la relation

$$\Gamma^i_{jk}(x, y) = \Gamma^i_{jk}(x)\, \forall x, y \in TM. \tag{1.4}$$

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1.2 Construction du fibré vectoriel π^*TM

Définition 1.2. *Soient M une variété différentiable de dimension n et $TM\backslash 0$ le « Slit tangent bundle » de M. La projection naturelle $\pi : TM\backslash 0 \to M$ « pull-back » l'espace π^*TM au dessus de $TM\backslash 0$.*
π^*TM est le fibré vectoriel dont le fibré de base est le fibré tangent $TM\backslash 0$.

$$\pi^*TM := \{(x, y, v) : (x, y) \in TM\backslash 0 \text{ et } v \in T_\pi(x, y)M\}.$$

Les fibres de π^*TM sont des copies de l'espace T_xM. On a $\pi^*TM|_{(x,y)} \cong T_xM$.

Définition 1.3. *Le « pull-back » du fibré cotangent π^*T^*M est un fibré vectoriel dual du fibré π^*TM et dont les fibres sont des copies de T_x^*M. Précisement, on a :*
$\pi^*T^*M|_{(x,y)} \cong T_x^*M$.

1.3 Quelques objets fondamentaux du fibré π^*TM

— Soient $(x^i)_{i=1,\cdots,n}$ les coordonnées sur un ouvert $U \subset M$ et $(x^i, y^i)_{i=1,\cdots,n}$ les coordonnées locales sur un ouvert $\pi^{-1}(U) \subset TM$. Les x^i produisent des sections de bases $\left\{\frac{\partial}{\partial x^i}\right\}$ et $\{dx^i\}$ respectivement pour TM et pour T^*M.
Puisque les fibres du fibré tangent pulled-back sont identiques à TM pendant que les fibres de π^*T^*M sont identiques à T_x^*M, alors $\left\{\frac{\partial}{\partial x^i}\right\}$ et $\{dx^i\}$ sont respectivement les sections de bases pour les fibrés π^*TM et π^*T^*M.
Avec ces sections de base, on peut définir quelques objets géometriques finsleriens associés au fibré π^*TM et à son dual π^*T^*M.

— Le fibré vectoriel π^*TM admet une métrique riemannienne naturelle $g := g_{ij}(x, y)\, dx^i \otimes dx^j$. C'est un tenseur fondamental dont les éléments sont définis dans (1.1).Ce dernier est une section symétrique du fibré $\pi^*T^*M \otimes \pi^*T^*M$.

— Un autre objet très important est le tenseur de Cartan $\mathcal{A} := \mathcal{A}_{ijk}(x, y)\, dx^i \otimes dx^j \otimes dx^k$ où

$$\mathcal{A}_{ijk} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} \tag{1.5}$$

$\mathcal{A}$ est une section symétrique du fibré $\pi^*T^*M \otimes \pi^*T^*M \otimes \pi^*T^*M$.

Lemme 1.2. *Soit F une métrique finslérienne sur une variété différentiable M, alors :*

1. $\frac{\partial g^{ij}}{\partial y^k} = -\frac{2}{F}\mathcal{A}_k^{ij}$ *où g^{ij} sont des composantes de l'inverse de g et $\mathcal{A}_k^{ij} = g^{il}g^{jm}\mathcal{A}_{lmk}$*
2. $\mathcal{A}_{ijk} = \mathcal{A}_{jik} = \mathcal{A}_{ikj}$
3. $y^i\mathcal{A}_{ijk} = 0.$

Lemme 1.3 (De Deicke)**.** *Une métrique finslérienne est riemannienne si le tenseur de cartan est nul.*

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1.4 Connexion non linéaire sur le Slit tangent bundle $TM\backslash 0$

Les composantes g_{ij} du tenseur fondamental g, sont des fonctions différentiables sur $TM\backslash 0$. On peut montrer, grâce au **Lemme 1.1 (Euler)** qu'elles sont invariantes sous la transformation $y \mapsto cy\ \forall c \in \mathbb{R}_0$. Elles sont utilisées pour définir les symboles de christoffel formels de la seconde forme définis comme suit :

$$\gamma^i_{jk} := \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \tag{1.6}$$

Les γ^i_{jk} déterminent les coefficients d'une connexion non-linéaire sur $TM\backslash 0$ donnée par :

$$N^i_j = \gamma^i_{jk} y^k - \frac{1}{F} \mathcal{A}_{jkl} g^{il} \gamma^k_{rs} y^r y^s, \forall i, j, k, r, s = 1, \cdots, n\ . \tag{1.7}$$

Par conséquent, sur $TM\backslash 0$,les bases dont on travaille avec sont $\left\{ \frac{\delta}{\delta x^i}, F\frac{\delta}{\delta y^i} \right\}_{i=1,\cdots,n}$ et son dual $\left\{ dx^i, \frac{\delta y^i}{F} \right\}_{i=1,\cdots,n}$ au lieu de $\left\{ \frac{\partial}{\partial x^i}, F\frac{\partial}{\partial y^i} \right\}_{i=1,\cdots,n}$ et son dual $\left\{ dx^i, \frac{dy^i}{F} \right\}_{i=1,\cdots,n}$.
Les facteurs F et $\frac{1}{F}$ sont introduits pour rendre tous les objets géométriques qui sont invariants sous la transformation $y \mapsto cy, (c > 0)$. De plus, le fibré tangent de $TM\backslash 0$ peut se décomposer en une somme directe de la partie horizontale $\mathcal{H}$ engendrée par $\left\{ \frac{\delta}{\delta x^i} \right\}_{i=1,\cdots,n}$ et la partie verticale $\mathcal{V}$ engendrée par $\left\{ F\frac{\partial}{\partial y^i} \right\}_{i=1,\cdots,n}$.

En d'autres termes, on a :

$$\begin{aligned} T\left(TM\backslash 0\right) &= vect\left\{ \frac{\delta}{\delta x^i} \right\} \oplus vect\left\{ F\frac{\delta}{\delta y^i} \right\} && (1.8)\\ &= \mathcal{H} \oplus \mathcal{V}. && (1.9) \end{aligned}$$

Avec les coefficients d'une connexion non-linéaire N^i_j, on peut définir une forme différentielle $\theta, \theta : \pi^*TM \to T\left(TM\backslash 0\right)$ par : $\theta := \frac{\delta}{\delta x^i} \otimes \frac{1}{F}\left(dy^i + N^i_j dx^j\right)$. A travers la décomposition de $T\left(TM\backslash 0\right)$, évoquée en (1.8) et en (1.9), $TM\backslash 0$ admet une connexion non-linéaire appelée **connexion d'Ehresmann**.

1.5 Connexiond linéaire sur l'espace π^*TM

Définition 1.4. [*16*] *Une connexion linéaire sur le fibré vectoriel π^*TM au dessus du fibré $TM\backslash 0$, est une application*

$$\bar{\nabla} : \mathcal{X}\left(TM\backslash 0\right) \times \Gamma\left(\pi^{*}TM\right) \rightarrow \Gamma\left(\pi^{*}TM\right)$$
$$(X, \xi) \mapsto \bar{\nabla}_{X}\xi$$

telle que $\forall f \in C^{\infty}\left(TM\backslash 0, \mathbb{R}\right)$, *satisfaisant aux propriétés suivantes :*

(i) $\bar{\nabla}_{X+Y}\xi = \bar{\nabla}_{X}\xi + \bar{\nabla}_{Y}\xi$

(ii) $\bar{\nabla}_{fX}\xi = f\bar{\nabla}_{X}\xi$

(iii) $\bar{\nabla}_{X}\left(f\xi\right) = X\left(f\right)\xi + f\bar{\nabla}\xi$

(iv) $\bar{\nabla}_{X}\left(\xi+\eta\right)$=$\bar{\nabla}_{X}\xi + \bar{\nabla}_{X}\eta$.

Remarque 1.2. *1.* $\bar{\nabla}_{X}\xi$ *est appelée derivée covariante de* ξ *dans la direction de* X *.*

2. *En coordonnées locales* $\left(x^{i}\right)_{i=1,\cdots,n}$*dans* M, *on a :*
   $\bar{\nabla}_{\frac{\delta}{\delta x^{i}}}\dfrac{\partial}{\partial x^{j}} = \Gamma_{ij}^{k}\dfrac{\partial}{\partial x^{k}}$ *où les* Γ_{ij}^{k} *sont des coefficients de* $\bar{\nabla}$.

1.6 Connexion de Finsler-Ehresmann

Considérons une application différentielle π_{*} de la submersion $\pi : TM\backslash 0 \rightarrow M$:$\pi(x, y) \mapsto x$. Le sous-espace vectoriel de $T\left(TM\backslash 0\right)$ est définie par $\mathcal{V} := Ker\left(\pi_{*}\right)$ et est localement engendré par l'ensemble $\left\{F\dfrac{\partial}{\partial y^{i}}, n \geq i \geq 1\right\}$ sur chaque $\pi^{-1}(U) \subset TM\backslash 0$.

Un sous-espace horizontal $\mathcal{H}$ de $T\left(TM\backslash 0\right)$ est par définition chaque complémentaire de $\mathcal{V}$. Les fibrés $\mathcal{H}$ et $\mathcal{V}$ donnent un découpage lisse [14]

$$T(TM\backslash 0) = \mathcal{H} \oplus \mathcal{V}. \tag{1.10}$$

Une connexion d'Ehresmann est une selection d'un sous-espace horizontal $\mathcal{H}$ de $T\left(TM\backslash 0\right)$. Le fibré $\mathcal{H}$ peut être défini canoniquement [9]à partir de l'équation géodésique.

Définition 1.5. *soit* π *la projection restreinte définie par* $\pi : TM\backslash 0 \rightarrow M$.

(1) une connexion de Finsler-Ehresmann de π *est le sous fibré* $\mathcal{H}$ *de* $T\left(TM\backslash 0\right)$ *défini par*

$$\mathcal{H} : Ker\left(\theta\right), \tag{1.11}$$

où $\theta : T\left(TM\backslash 0\right) \rightarrow \pi^{*}TM$ *est le morphisme défini par*

$$\theta = \frac{\partial}{\partial x^{i}} \otimes \frac{1}{F}\left(dy^{i} + N_{j}^{i}dx^{j}\right) \tag{1.12}$$

avec $N_{j}^{i}(x, y) := \dfrac{\partial G^{i}(x, y)}{\partial y^{j}}$ *pour*

$$G^{i}(x, y) := \frac{1}{4}g^{jl}(x, y)\left[\frac{\partial g_{jl}}{\partial x^{k}}(x, y) + \frac{\partial g_{kl}}{\partial x^{j}}(x, y) - \frac{\partial g_{jk}}{\partial x^{l}}(x, y)\right]y^{j}y^{k}. \tag{1.13}$$

(2) La forme $\theta : T\left(TM\backslash 0\right) \to \pi^{*}TM$ induit une application linéaire

$$\theta|_{(x,y)} : T_{(x,y)}TM\backslash 0 \to T_xM, \tag{1.14}$$

pour chaque point $(x,y) \in TM\backslash 0$ où $x = \pi\left(x,y\right)$.
Le relevement vertical d'une section ξ de $\pi_{*}TM$ est l'unique section $\mathrm{v}\left(\xi\right)$ de $T\left(TM\backslash 0\right)$ telle que pour tout $(x,y) \in TM\backslash 0$,

$$\pi_{*}\left(\mathrm{v}\left(\xi\right)\right)|_{(x,y)} = \mathcal{O}_{(x,y)} \textit{ et } \theta\left(\mathrm{v}\left(\xi\right)\right)|_{(x,y)} = \xi_{(x,y)}. \tag{1.15}$$

(3) La projection différentielle $\pi_{*} : T\left(TM\backslash 0\right) \to \pi^{*}TM$ induit une application linéaire

$$\pi_{*}|_{(x,y)} : T_{(x,y)}TM\backslash 0 \to T_xM, \tag{1.16}$$

pour chaque point $(x,y) \in TM\backslash 0$; où $x \in M = \pi\left(x,y\right)$.
Le relevement horizontal d'une section ξ de $\pi^{*}TM$ est l'unique section $\mathrm{h}\left(\xi\right)$ de $T\left(TM\backslash 0\right)$ telle que pour tout $(x,y) \in TM\backslash 0$,

$$\pi_{*}\left(\mathrm{h}(\xi)\right)|_{(x,y)} = \xi_{(x,y)} \textit{ et } \theta\left(\mathrm{h}\left(\xi\right)\right) = \mathcal{O}_{(x,y)}. \tag{1.17}$$

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Chapitre 2

Courbures berwaldiennes

2.1 Champ de Tenseurs Finslerien

Définition 2.1. *[16]Un champ de tenseurs finslerien T de type $(q, 0, p_1, p_2)$ sur $TM\backslash 0$ est une section C^∞ du fibré vectoriel*

$$\underbrace{\pi^* T^* M \otimes \cdots \otimes \pi^* T^* M}_{p_1 - facteurs} \otimes \underbrace{T^* (TM\backslash 0) \otimes \cdots \otimes T^* (TM\backslash 0)}_{p_2 - facteurs} \otimes^q \pi^* TM. \tag{2.1}$$

C'est-à-dire que $T : \pi^* T^* M \otimes \cdots \otimes \pi^* T^* M \otimes T^* (TM\backslash 0) \otimes \cdots \otimes^q T^* M \to C^\infty (TM \times \cdots, \mathbb{R})$ est une application multilinéaire.

Remarque 2.1. *Dans une carte locale,*
$T = T^{k_1 \cdots k_q}_{i_1 \cdots i_p, j_1 \cdots j_{p_2}} \partial_{k_1} \otimes \cdots \otimes \partial_{k_q} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{p_2}}$
où $\left(\partial_{x_1} \otimes \cdots \otimes \partial_{x_q} \otimes dx^i \otimes \cdots \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_{p_2}}\right)$ *avec* $k \in \{1, \cdots, n\}\, q$*,*$i \in \{1, \cdots, n\}\, p_1$*,* $j \in \{1, \cdots, n\}\, p_2$ *est une section de base de tenseur et les* $\partial_{k_r} := \dfrac{\partial}{\partial x^{k_r}}$ *ainsi que* ε^{j_s} *sont respectivement les sections de base pour* $\pi^* TM$ *et* $T^* (TM\backslash 0)$ *dual de* $T (TM\backslash 0)$.

Exemple 2.1.
1. *Le tenseur g est de type $(0, 0, 2, 0)$*
2. *La forme de Finsler-Ehresmann est de type $(1, 0, 0, 1)$.*

La connexion de Chern sur $\pi^* TM$ est définie par le lemme suivant :

Lemme 2.1. *Considérons (M, F) une variété finslerienne et g son tenseur fondamental. Il existe une unique connexion ∇ sur le fibré vectoriel $\pi^* TM$ tel que pour $X, Y \in \mathcal{X} (TM\backslash 0)$ et $\forall \xi, \eta \in \pi^* TM$, on a les propriétés suivantes :*

1. $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_* [X, Y]$
2. $X (g (\xi, \eta)) = g (\nabla_X \xi, \eta) + g (\xi, \nabla_X \eta) + 2\mathcal{A} (\theta (X), \xi, \eta)$.
   Où $\mathcal{A} := \dfrac{F}{2} \dfrac{\partial}{\partial y^k} g_{ij} dx^i \otimes dx^j \otimes dx^k$ *est le tenseur de Cartan et θ est la forme d'Ehresmann définie dans* (1.12).

Les coefficients de cette connexion sont

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{kl}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right) \tag{2.2}$$

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où

— Γ^i_{jk} est le symbole de Christoffel de la connexion de Chern,

—

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} \tag{2.3}$$

$$= \mathrm{h}\left(\frac{\partial}{\partial x^i}\right), \tag{2.4}$$

avec $N_j^i = \Gamma^i_{jk} y^k$.

A l'instar de la connexion de Levi-civita en géométrie riemannienne, la connexion de Chern est une connexion linéaire ne possédant pas de torsion, mais n'obéit qu'à des conditions de g-compatibilités restreintes.

2.2 Opérateurs différentiels fondamentaux sur $TM\backslash 0$

2.2.1 Gradient du fibré vectoriel π^*TM

Définition 2.2. *[17] Soient F une métrique finslérienne sur une variété différentiable M et $f \in C^\infty(TM\backslash 0)$. Le gradient horizontal de f est la section π^*TM définie par*

$$\nabla^h f = g^{ij} \frac{\delta f}{\delta x^i} \frac{\partial}{\partial x^j} \quad . \tag{2.5}$$

Le gradient vertical de f est la section π^*TM définie par

$$\nabla^v f = g^{ij} F \frac{\delta f}{\delta y^i} \frac{\partial}{\partial x^j} \quad . \tag{2.6}$$

2.2.2 Divergence du fibré vectoriel π^*TM

Définition 2.3. *[17] Soit $\xi \in \Gamma(\pi^*TM)$, on définit la divergence horizontale par*

$$div^h \xi = trace_g \left(\eta \mapsto \nabla^h_\eta \xi\right), \tag{2.7}$$

et la divergence verticale par

$$div^v \xi = trace_g \left(\eta \mapsto \nabla^h_\eta \xi\right). \tag{2.8}$$

Où g est le tenseur fondamental associé à F et ∇ est la connexionn de Chern.
Dans les sections de base $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\cdots,n}$ du fibré π^*TM, on a

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$$div^h\xi = g^{ij}g\left(\nabla_{\frac{\delta}{\delta x^i}}\xi, \frac{\partial}{\partial x^j}\right) \text{ et } div^v\xi = g^{ij}g\left(\nabla_{F\frac{\partial}{\partial y^i}}, \frac{\partial}{\partial x^j}\right). \tag{2.9}$$

La divergence verticale de ξ est une fonction C^∞ sur $TM\backslash 0$. on obtient

$$div^v\xi = F\frac{\partial \xi^i}{\partial y^i} \quad . \tag{2.10}$$

2.2.3 Les Laplaciens des fonctions C^∞sur $TM\backslash 0$

Définition 2.4. *[17] Soient F une métrique finslérienne sur une variété différentiable M et $f \in C^\infty(TM\backslash 0)$. Le Laplacien horizontal $\Delta^h f$ de f est défini par*

$$\Delta^h f = div^h(\nabla^h f), \tag{2.11}$$

et le Laplacien vertical $\Delta^h f$ de f est défini par

$$\Delta^v f = div^v(\nabla^v f). \tag{2.12}$$

2.3 Métrique de Berwald

Définition 2.5. *Soit F_t une métrique finslérienne sur une variété M de dimension n. Soit $(x, y) \in TM$, on dit que F_t est une métrique berwaldienne, si pour une coordonnée locale $(x^i, y^i)_{i,\cdots,n}$ dans $TM\backslash 0$, les symboles de Christoffel Γ^l_{ij} de la connexion de Chern ne dépendent pas de la direction y dans TM.*

La donnée du couple (M, F_t) est un espace berwaldien du tenseur fondamental $g = \Big(g_{ij}(t, x, y)\Big)$.

Exemple 2.2. *Toutes les métriques riemanniennes et toutes les métriques localement minkowskiennes sont des exemples des métriques de Berwald.*

(1) *Pour les métriques riemanniennes, $\Gamma^i_{jk} = \gamma^i_{jk}$. En particulier, les fonctions Γ^i_{jk} sont indépendantes de y*

(2) *Pour les métriques localement minkowskiennes, dans un voisinage de U d'un point $x \in M$, les fonctions Γ^k_{ij} s'annulent identiquement.*

2.4 Courbure complète berwaldienne

Définition 2.6. *La courbure complète associée à la connexion de chern ∇ sur le fibré vectoriel π^*TM au dessus de la variété $TM\backslash 0$ est l'application*
$\phi : \mathcal{X}\,(TM\backslash 0) \times \mathcal{X}\,(TM\backslash 0) \times \Gamma\,(\pi^*TM) \to \Gamma\,(\pi^*TM)$

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$$(X, Y, \xi) \mapsto \phi(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi.$$

Par le relation (1.10), on obtient $\nabla_X = \nabla_{\hat{X}} + \nabla_{\check{X}}$ où $X = \hat{X} + \check{X}$ avec $\hat{X} \in \Gamma(\mathcal{H})$ et $\check{X} \in \Gamma(\mathcal{V})$.

On peut alors définir la courbure complète de ∇ comme suit :

$$\begin{aligned} \Phi(\xi, \eta, X, Y) &= g(\phi(X, Y)\xi, \eta) && (2.13)\\ &= g\left(\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi, \eta\right). && (2.14) \end{aligned}$$

Comme $X = \hat{X} + \check{X}$,$Y = \hat{Y} + \check{Y}$, (2.14) devient :
$\Phi(\xi, \eta, X, Y) = g\left[(\nabla_{\hat{X}} + \nabla_{\check{X}})(\nabla_{\hat{Y}} + \nabla_{\check{Y}})\xi - (\nabla_{\hat{Y}} + \nabla_{\check{Y}})(\nabla_{\hat{X}} + \nabla_{\check{X}})\xi - \nabla_{\left[\hat{X}+\check{X},\hat{Y}+\check{Y}\right]}\xi, \eta\right]$
Or, puisque ∇ est linéaire, on a :

$$\begin{aligned} \nabla_{\left[\hat{X}+\check{X},\hat{Y}+\check{Y}\right]}\xi &= \nabla_{\left(\hat{X}+\check{X}\right)\left(\hat{Y}+\check{Y}\right)-\left(\hat{Y}+\check{Y}\right)\left(\hat{X}+\check{X}\right)}\xi \\ &= \nabla_{\hat{X}\hat{Y}+\hat{X}\check{Y}+\check{X}\hat{Y}-\hat{Y}\hat{X}-\hat{Y}\check{X}-\check{Y}\hat{X}-\check{Y}\check{X}+\check{X}\check{Y}}\xi \\ &= \nabla_{\left[\hat{X},\hat{Y}\right]}\xi + \nabla_{\left[\hat{X},\check{Y}\right]}\xi + \nabla_{\left[\check{X},\hat{Y}\right]}\xi + \nabla_{\left[\check{X},\check{Y}\right]}\xi \quad . \end{aligned}$$

Ainsi,

$$\begin{aligned} \Phi(\xi, \eta, X, Y) &= g(\nabla_{\hat{X}}\nabla_{\hat{Y}}\xi + \nabla_{\hat{X}}\nabla_{\check{Y}}\xi + \nabla_{\check{X}}\nabla_{\hat{Y}}\xi + \nabla_{\check{X}}\nabla_{\check{Y}}\xi - \nabla_{\hat{X}}\nabla_{\hat{X}}\xi - \nabla_{\hat{Y}}\nabla_{\check{X}}\xi \\ &\quad -\nabla_{\check{Y}}\nabla_{\hat{X}}\xi - \nabla_{\check{Y}}\nabla_{\check{X}}\xi - \nabla_{\left[\hat{X},\hat{Y}\right]}\xi - \nabla_{\left[\hat{X},\check{Y}\right]}\xi - \nabla_{\left[\check{X},\hat{Y}\right]}\xi - \nabla_{\left[\check{X},\check{Y}\right],\xi}\xi) \quad . \end{aligned} \tag{2.15}$$

Comme $\phi(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$

$$\begin{aligned} \Phi(\xi, \eta, X, Y) &= g\left[\phi(\hat{X}, \hat{Y})\xi + \phi(\hat{X}, \check{Y})\xi + \phi(\check{X}, \hat{Y})\xi + \phi(\check{X}, \check{Y})\xi, \eta\right] \\ &= g\left(\phi(\hat{X}, \hat{Y})\xi, \eta\right) + g\left(\phi(\hat{X}, \check{Y})\xi, \eta\right) + g\left(\phi(\check{X}, \hat{Y})\xi, \eta\right) + g\left(\phi(\check{X}, \check{Y})\xi, \eta\right) \\ &= R(\xi, \eta, X, Y) + P(\xi, \eta, X, Y) + Q(\xi, \eta, X, Y). \end{aligned}$$

Avec
$R(\xi, \eta, X, Y) = g\left(\phi(\hat{X}, \hat{Y})\xi, \eta\right)$: La courbure horizontale
$P(\xi, \eta, X, Y) = g\left(\phi(\hat{X}, \check{Y})\xi, \eta\right) + g\left(\phi(\check{X}, \hat{Y})\xi, \eta\right)$: La courbure mixte
$Q(\xi, \eta, X, Y) = g\left(\phi(\check{X}, \check{Y})\xi, \eta\right)$: La courbure verticale.

En particulier, si ∇ est la connexion de Chern, la $Q-$ courbure s'annule.

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Dans un repère local, les composantes de la courbure de Chern sont :

$$\begin{aligned}\Phi(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l) &= R(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_{kl})+P(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_{kl}) \\ &= \left(\frac{\delta\Gamma^s_{il}}{\delta x^k}-\frac{\delta\Gamma^s_{ik}}{\delta x^l}\right)g_{js}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}-F\frac{\partial\Gamma^s_{ik}}{\partial y^l} \quad (2.16)\end{aligned}$$

$$\text{avec } \partial_i=\frac{\partial}{\partial x^i}\in\Pi^*TM,\ \hat{\partial}_k:=\frac{\delta}{\delta x^k}\in\mathcal{H} \text{ et } \check{\partial}_k:=F\frac{\partial}{\partial y^k}\in\mathcal{V}.$$

Si F_t est une métrique berwaldienne, alors d'après la **Définition 2.5**, la courbure associée à la connexion de Chern est :

$$\Phi_{ijkl}=\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}\ , \quad (2.17)$$

avec $\Phi_{ijkl}=\Phi(\partial_i,\partial_j,\hat{\partial}_k+\check{\partial}_k,\hat{\partial}_l+\check{\partial}_l)$.

Lemme 2.2. *L'identité de Berwaldienne de Bianchi définie par* $\forall\xi,\eta\in\Gamma\left(\pi^*TM\right)$ *et* $X,Y,Z\in\xi\left(TM\backslash 0\right)$ *alors*

$$\left(\nabla_Z\Phi\right)(\xi,\eta,X,Y)+\left(\nabla_X\Phi\right)(\xi,\eta,Y,Z)+\left(\nabla_Y\Phi\right)(\xi,\eta,Z,X)=0\ . \quad (2.18)$$

Démonstration. Ce lemme est prouvé à partir de la symétrie de ∇ et de l'identité de Jacobi. En posant que $(\star)=\left(\nabla_Z\Phi\right)(\xi,\eta,X,Y)+\left(\nabla_X\Phi\right)(\xi,\eta,Y,Z)+\left(\nabla_Y\Phi\right)(\xi,\eta,Z,X)$, on aura

$$\begin{aligned}(*) \;=\;& Z\Phi(\xi,\eta,X,Y)-\Phi(\nabla_Z\xi,\eta,X,Y)-\Phi(\eta,\nabla_Z\xi,X,Y)-\Phi(\xi,\eta,\nabla_ZX,Y) \\ &-\Phi(\xi,\eta,X,\nabla_ZY)+X\Phi(\xi,\eta,Y,Z)-\Phi(\nabla_X\xi,\eta,Y,Z)-\Phi(\xi,\nabla_X\eta,Y,Z) \\ &-\Phi(\xi,\eta,\nabla_XY,Z)-\Phi(\xi,\eta,Y,\nabla_XZ)+Y\Phi(\xi,\eta,Z,X)-\Phi(\nabla_Y\xi,\eta,Z,X) \\ &-\Phi(\xi,\nabla_Y\xi,Z,X)-\Phi(\xi,\eta,\nabla_YZ,X)-\Phi(\xi,\eta,Z,\nabla_YX)\ . \quad (2.19)\end{aligned}$$

Par la symétrie de ∇,

$$\begin{aligned}\Phi(\xi,\eta,\nabla_XY,Z) &= -\Phi(\xi,\eta,Z,\nabla_XY) \quad (2.20)\\ &= -\Phi(\xi,\eta,Z,-\nabla_YX) \quad (2.21)\\ &= \Phi(\xi,\eta,Z,\nabla_YX)\ . \quad (2.22)\end{aligned}$$

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Par substitution de (2.22) dans (2.19), on a

$$\begin{aligned}(*) &= Z\Phi(\xi,\eta,X,Y)-\Phi(\nabla_Z\xi,\eta,X,Y)-\Phi(\xi,\nabla_Z\eta,X,Y)+\Phi(\xi,\eta,Y,\nabla_Z X)\\ &+\Phi(\xi,\eta,X,\nabla_Y Z)+X\Phi(\xi,\eta,X,Z)-\Phi(\nabla_X\xi,\eta,X,Z)-\Phi(\xi,\nabla_X\eta,Y,Z)\\ &+\Phi(\xi,\eta,Z,\nabla_X Y)+\Phi(\xi,\eta,Y,\nabla_Z X)+Y\Phi(xi,\eta,Z,X)-\Phi(\nabla_Y\xi,\eta,Z,X)\\ &-\Phi(\xi,\nabla_Y\eta,Z,X)+\Phi(\xi,\eta,X,\nabla_Y Z)+\Phi(\xi,\eta,Z,\nabla_X Y). \end{aligned} \tag{2.23}$$

Du fait que $\Phi(\xi,\eta,X,Y)=g\Big(\phi(X,Y)\xi,\eta\Big)$, alors (2.23) prend la forme suivante

$$\begin{aligned}(*) &= Zg\Big(\phi(X,Y)\xi,\eta\Big)-g\Big(\phi(X,Y)\nabla_Z\xi,\eta\Big)-g\Big(\phi(X,Y)\xi,\nabla_Z\eta\Big)+g\Big(\phi(Y,\nabla_Z X)\xi,\eta\Big)\\ &+g\Big(\phi(X,\nabla_Y Z)\xi,\eta\Big)+Xg\Big(\phi(Y,Z)\xi,\eta\Big)-g\Big(\phi(Y,Z)\nabla_x\xi,\eta\Big)-g\Big(\phi(Y,Z)\xi,\nabla_X\eta\Big)\\ &+g\Big(\phi(Z,\nabla_X Y)\xi,\eta\Big)+g\Big(\phi(Y,\nabla_Z X)\xi,\eta\Big)+Yg\Big(\phi(Z,X)\xi,\eta\Big)-g\Big(\phi(Z,X)\nabla_Y\xi,\eta\Big)\\ &-g\Big(\phi(Z,X)\xi,\nabla_Y\eta\Big)+g\Big(\phi(X,\nabla_Y Z)\xi,\eta\Big)+g\Big(\phi(Z,X)\xi,\nabla_Y\eta\Big).\end{aligned}$$

$$\begin{aligned}(*) &= 2\underbrace{\Big[g\Big(\phi(Z,\nabla_X Y)\xi,\eta\Big)+g\Big(\phi(X,\nabla_Y Z)\xi,\eta\Big)+g\Big(\phi(Y,\nabla_Z X)\xi,\eta\Big)\Big]}_{(**)}+Zg\Big(\phi(X,Y)\xi,\eta\Big)\\ &-g\Big(\phi(X,Y)\nabla_Z\xi,\eta\Big)-g\Big(\phi(X,Y)\xi,\nabla_Z\eta\Big)+Xg\Big(\phi(Y,Z)\xi,\eta\Big)-g\Big(\phi(Y,Z)\nabla_X\xi,\eta\Big)\\ &-g\Big(\phi(Y,Z)\nabla_X\xi,\eta\Big)-g\Big(\phi(Y,Z)\xi,\nabla_X\xi\Big)+Yg\Big(\phi(Z,X)\xi,\eta\Big)-g\Big(\phi(Z,X)\nabla_Y\xi,\eta\Big)\\ &-g\Big(\phi(Z,X)\xi,\nabla_Y\eta\Big).\end{aligned}$$

D'après l'identité de Jacobi $(**)=0$ et la définition de $(\nabla_X g)(X,Y)=Xg(Y,Z)-g(\nabla_X Y,Z)-g(Y,\nabla_X Z)=0$, avec ∇ une connexion sans torsion

$$\begin{aligned}(*) &= (\nabla_Z g)\Big(\phi(X,Y)\xi,\eta\Big)+(\nabla_X g)\Big(\phi(Y,Z)\xi,\eta\Big)+(\nabla_Y g)\Big(\phi(Z,X)\xi,\eta\Big)\\ &= 0.\end{aligned}$$

□

En contractant deux fois l'équation (2.18) écrite en coordonnées locales , on a :

$$\frac{1}{2}\nabla_j Scal=\nabla^i Ric(\partial_i,\hat{\partial}_j). \tag{2.24}$$

2.5 Tenseurs de courbures berwaldiens

D'après la connexion de Chern, on obtient les définitions du tenseur de Ricci berwaldien et de la courbure scalaire berwaldienne suivantes :

Définition 2.7. *[16]*

*(1) Le tenseur de Ricci berwaldien **Ric** de (M, F) est défini par*

$$\boldsymbol{Ric}(\xi, X) := trace_g\Big[\eta \mapsto R\Big(X, \mathrm{h}(\eta) + \mathrm{v}(\eta)\Big)\xi\Big]. \tag{2.25}$$

Locallement, on a

$$\begin{aligned} \boldsymbol{Ric}(\partial_i, \hat{\partial}_k) &= \frac{\partial \Gamma^l_{il}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{il}\Gamma^l_{ks} && (2.26) \\ &= g^{jl}\Phi_{ijkl}. && (2.27) \end{aligned}$$

*(2) La courbure scalaire berwaldien **Scal** de (M, F) est définie par*

$$\boldsymbol{Scal} := trace_g(\boldsymbol{Ric}), \underline{g} := \pi^* g. \tag{2.28}$$

Localement, on a

$$\begin{aligned} \boldsymbol{Scal}(\partial_i, \hat{\partial}_k + \check{\partial}_k) &= \boldsymbol{Ric}_{ik} g^{ik} && (2.29) \\ &= \left(\frac{\partial \Gamma^l_{il}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{il}\Gamma^l_{ks}\right) g^{ik} \quad . && (2.30) \end{aligned}$$

Dans ce chapitre, nous avons évoqué les notions de la courbure associée à la connexion de Chern, ainsi que les notions du tenseur de Ricci berwaldien et de la courbure scalaire berwaldienne sur (M, F_t) très utiles dans le chapitre suivant.
Ce sont ces notions qui vont nous aider à trouver les équations d'évolution hyperbolique sur les espaces berwaldiens.

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Chapitre 3

Les équations d'évolution hyperbolique

Dans ce chapitre, nous dérivons les équations d'évolution des courbures dans les espaces berwaldiens le long du flux géométrique hyperbolique.

3.1 Le flux géométrique hyperbolique

Dans les espaces rimanniens, le flux géométrique hyperbolique est un système d'équations hyperbolique sur les métriques qui a été introduit par Kong et Liu [11].

Condidérons une variété M, une famille à un paramètre $\{F_t\}_{t\in[0,\lambda]}$ de la métrique finslérienne sur M et $\{g_t\}_{t\in[0,\lambda]}$ la famille des tenseurs fondamentaux associée à la famille $\{F_t\}$ (voir [2]).

Définition 3.1. *On appelle flux géométrique hyperbolique une déformation de la courbure de Ricci berwaldienne, l'évolution*

$$\frac{\partial^2}{\partial t^2} g(t) := -2\boldsymbol{Ric}_{F(t)}. \tag{3.1}$$

On dit que $g(t)$ est la solution du flux géométrique si elle satisfait (3.1). cette dernière existe dans des cas particuliers notamment dans les espaces riemanniens ([11],[6],[7],[12],[5]).

Le flux géométrique hyperbolique est une équation d'évolution sur la métrique $g_{ij}(t,x,y)$ qui implique une équation d'onde non linéaire pour le tenseur de courbure de Ricci et la courbure scalaire.

3.2 Equations d'évolution hyperbolique de la courbure totale berwaldienne

Lemme 3.1. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t,x,y)\Big)$. Alors sous un paramètre t de la courbure scalaire définie sur (M, F_t), l'élément g^{ij} vérifie l'équation d'évolution suivante :*

$$\frac{\partial}{\partial t} g^{jl} = -g^{jr} g^{ls} \frac{\partial g_{rs}}{\partial t}. \tag{3.2}$$

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Démonstration. Soit $g^{jr}g_{rs} = \delta_s^j$.
En dérivant, on a
$\frac{\partial}{\partial t}\left(g^{jr}g_{rs}\right) = 0$
$\Leftrightarrow \frac{\partial}{\partial t}\left(g^{jr}\right)g_{rs} + \frac{\partial}{\partial t}\left(g_{rs}\right)g^{jr} = 0$
$\Leftrightarrow \frac{\partial}{\partial t}\left(g^{jr}\right)g_{rs} = -\frac{\partial}{\partial t}\left(g_{rs}\right)g^{jr}$
$\Leftrightarrow \frac{\partial}{\partial t}\left(g^{jr}\right)g_{rs} = -\frac{\partial}{\partial t}\left(g_{rs}\right)g^{jr}$,avec $r = l$
$\Leftrightarrow \frac{\partial}{\partial t}g^{jl} = -g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}$. □

Lemme 3.2. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t, x, y)\Big)$. Alors sous un paramètre t de la courbure scalaire définie sur (M, F_t), l'élément g^{ij} vérifie l'équation d'évolution hyperbolique suivante :*

$$\frac{\partial^2 g^{jl}}{\partial t^2} = \Big(g^{jr}g^{lp}g^{sq} + g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{jr}g^{ls}Ric_{rs} \quad . \tag{3.3}$$

Démonstration.

$$\begin{aligned}
\frac{\partial^2 g^{jl}}{\partial t^2} &= \frac{\partial}{\partial t}\left(-g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\Big(-g^{jr}g^{ls}\Big)\frac{\partial g_{rs}}{\partial t} + \Big(-g^{jr}g^{ls}\Big)\frac{\partial^2 g_{rs}}{\partial t^2} \\
&= -\left(g^{jr}\frac{\partial g^{ls}}{\partial t} + g^{ls}\frac{\partial g^{jr}}{\partial t}\right)\frac{\partial g_{rs}}{\partial t} - g^{jr}g^{ls}\frac{\partial^2 g_{rs}}{\partial t^2} \quad .
\end{aligned}$$

D'après le **Lemme 3.1**, on a $\frac{\partial}{\partial t}g^{ls} = -g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}$ et $\frac{\partial}{\partial t}g^{jr} = -g^{jp}g^{rq}\frac{\partial g_{pq}}{\partial t}$. Alors

$$\frac{\partial^2 g^{jl}}{\partial t^2} = \Big(g^{jr}g^{lp}g^{sq} + g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} - g^{jr}g^{ls}\frac{\partial^2 g_{rs}}{\partial t^2}. \tag{3.4}$$

En substituant (3.1) dans (3.4) , on a

$$\frac{\partial^2 g^{jl}}{\partial t^2} = \Big(g^{jr}g^{lp}g^{sq} + g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{jr}g^{ls}Ric_{rs}.$$

□

3.2.1 Evolution hyperbolique des symboles de Christoffel

Lemme 3.3. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t, x, y)\Big)$. Alors sous un paramètre t de Ricci d'un espace berwaldien (M, F_t), les symboles de Christoffel Γ_{il}^s vérifient l'équation d'évolution suivante :*

$$\frac{\partial}{\partial t}\Gamma^s_{il} = \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) - \frac{1}{2}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right). \tag{3.5}$$

Démonstration. En dérivant les symbols de Christoffel par rapport au paramètrs t, on obtient :

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^s_{il} &= \frac{\partial}{\partial t}\left[\frac{1}{2}g^{sj}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right)\right] \\ &= \frac{1}{2}g^{sj}\frac{\partial}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^{ji}} - \frac{\partial g_{il}}{\partial x^j}\right) + \frac{1}{2}\left(\frac{\partial}{\partial t}g^{sj}\right)\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right) \\ &= \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) + \frac{1}{2}(\frac{\partial}{\partial t}g^{sj})\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right).\end{aligned}$$

Avec $\frac{\partial g^{sj}}{\partial t} = -g^{sp}g^{jp}\frac{\partial g_{pq}}{\partial t}$, on a

$$\frac{\partial}{\partial t}\Gamma^s_{il} = \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) - \frac{1}{2}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right).$$

□

Proposition 3.1. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t,x,y)\Big)$. Alors sous un paramètre t de la courbure de Ricci définie sur (M, F_t), les symboles de Christoffel Γ^s_{il} vérifient l'équation d'évolution hyperbolique suivante :*

$$\begin{aligned}\frac{\partial^2\Gamma^s_{il}}{\partial t^2} &= \frac{1}{2}\left[\left(g^{sp}g^{jr}g^{qk} + g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\right]\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right) \\ &\quad -g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial x^{lj}})\right) \\ &\quad -g^{sj}\left(\frac{\partial}{\partial x^l}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jl}) - \frac{\partial}{\partial x^j}(Ric_{il})\right).\end{aligned} \tag{3.6}$$

Démonstration. En dérivant l'équation d'évolution des symboles de Christoffel par rapport au paramètre t, on obtient

$$\begin{aligned}
\frac{\partial^2\Gamma^s_{il}}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial t}\right)\\
&= \frac{1}{2}\frac{\partial g^{sj}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&\quad+\frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{ji}}{\partial t^2})+\frac{\partial}{\partial x^i}(\frac{\partial^2 g_{jl}}{\partial t^2})-\frac{\partial}{\partial x^j}(\frac{\partial^2 g_{il}}{\partial t^2})\right)\\
&\quad+\frac{1}{2}\frac{\partial^2 g^{sj}}{\partial t^2}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)+\frac{1}{2}\frac{\partial g^{sj}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&= \frac{1}{2}\frac{\partial^2 g^{sj}}{\partial t^2}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)+\frac{\partial g^{sj}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&\quad+\frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{ji}}{\partial t^2})+\frac{\partial}{\partial x^i}(\frac{\partial^2 g_{jl}}{\partial t^2})-\frac{\partial}{\partial x^j}(\frac{\partial^2 g_{il}}{\partial t^2})\right).
\end{aligned}$$

Avec

$$\frac{\partial g^{sj}}{\partial t} = -g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t} \text{ et } \frac{\partial^2 g^{sj}}{\partial t^2} = \left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq},$$

(d'après le **Lemme 3.1** et **Lemme 3.2**) et avec la **Définition 3.1**, on obtient

$$\begin{aligned}
\frac{\partial^2\Gamma^s_{il}}{\partial t^2} &= \frac{1}{2}\left[\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right]\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
&\quad-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&\quad-g^{sj}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right).
\end{aligned}$$

□

3.2.2 Evolution hyperbolique de la courbure totale

L'évolution de la courbure associée à la connexion de Chern définie dans (2.17) est la dérivée de Φ_{ijkl} par rapport au paramètre t. Dans ce cas, on obtient :

$$\begin{aligned}
\frac{\partial\Phi_{ijkl}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}\\
&\quad+\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t}.
\end{aligned} \tag{3.7}$$

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Et l'évolution hyperbolique de la courbure totale est donnée par l'expression suivante :

$$\begin{aligned}
\frac{\partial^2\Phi_{ijkl}}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\Phi_{ijkl}}{\partial t}\right) \\
&= \frac{\partial^2}{\partial t^2}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t} \\
&\quad +\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}+\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial^2 g_{js}}{\partial t^2} \\
&\quad +\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}+\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t} \\
&\quad +\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial^2 g_{jr}}{\partial t^2} \\
&= \frac{\partial^2}{\partial t^2}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr} \\
&\quad +(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial^2 g_{jr}}{\partial t^2}+2\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t} \\
&\quad +2\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t}+\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial^2 g_{js}}{\partial t^2}.
\end{aligned} \tag{3.8}$$

En substituant (3.1) dans (3.8) , on obtient la proposition suivante :

Proposition 3.2. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t, x, y)\Big)$. Alors sous un paramètre t de la courbure de Ricci définie sur (M, F_t), la courbure associée à la connexion de Chern vérifie l'équation d'évolution hyperbolique suivante :*

$$\begin{aligned}
\frac{\partial^2\Phi_{ijkl}}{\partial t^2} &= -2Ric_{js}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)-2Ric_{jr}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big) \\
&\quad +\frac{\partial^2}{\partial t^2}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\frac{\partial^2}{\partial t^2}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr} \\
&\quad +2\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}+2\frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\frac{\partial g_{jr}}{\partial t} \quad .
\end{aligned} \tag{3.9}$$

3.3 Equations d'évolution hyperbolique de la courbure de Ricci berwaldienne

Avec le rendement des calculs directs des équations d'évolution et des équations d'évolution hyperbolique de la courbure complète Φ_{ijkl} et par le fait que $Ric_{ik} = g^{jl}\Phi_{ijkl}$, si Φ_{ijkl} est constant par rapport à x, i.e $\Phi_{ijkl}(x,y) = \Phi_{ijkl}(t)$, alors

$$\frac{\partial}{\partial t} Ric_{ik} = \frac{\partial g^{jl}}{\partial t}\Phi_{ijkl} + g^{jl}\frac{\partial}{\partial t}\Phi_{ijkl} \ . \tag{3.10}$$

$$\begin{aligned} \frac{\partial^2 Ric_{ik}}{\partial t^2} &= \Phi_{ijkl}\frac{\partial^2 g^{jl}}{\partial t^2} + \frac{\partial g^{jl}}{\partial t}\frac{\partial \Phi_{ijkl}}{\partial t} + \frac{\partial g^{jl}}{\partial t}\frac{\partial \Phi_{ijkl}}{\partial t} + g^{jl}\frac{\partial^2 \Phi_{ijkl}}{\partial t^2} \\ &= g^{jl}\frac{\partial^2 \Phi_{ijkl}}{\partial t^2} + \Phi_{ijkl}\frac{\partial^2 g^{jl}}{\partial t^2} + 2\frac{\partial g^{jl}}{\partial t}\frac{\partial \Phi_{ijkl}}{\partial t}. \end{aligned} \tag{3.11}$$

En substituant (3.8), (3.7) et (2.17) dans (3.11), on obtient :

$$\begin{aligned} \frac{\partial^2 Ric_{ik}}{\partial t^2} &= g^{jl}\Bigg[-2Ric_{js}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2Ric_{jr}\Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big) \\ &+ \frac{\partial^2}{\partial t^2}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) g_{js} + \frac{\partial^2}{\partial t^2}\Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big) g_{jr} \\ &+ 2\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t} + 2\frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big)\frac{\partial g_{jr}}{\partial t}\Bigg] \\ &+ 2\frac{\partial g^{jl}}{\partial t}\Bigg[\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) g_{js} + \left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t} \\ &+ \frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big) g_{jr} + \Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big)\frac{\partial g_{jr}}{\partial t}\Bigg] \\ &+ \frac{\partial^2 g^{jl}}{\partial t^2}\left(\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) g_{js} + \Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big) g_{jr}\right). \end{aligned} \tag{3.12}$$

D'après les **Lemmes 3.1 et 3.2**, l'expression (3.12) devient

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$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= g^{jl}\Bigg[-2Ric_{js}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)-2Ric_{jr}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\\
&+\frac{\partial^2}{\partial t^2}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\frac{\partial^2}{\partial t^2}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr}\\
&+2\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}+2\frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\frac{\partial g_{jr}}{\partial t}\Bigg]\\
&-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\Bigg[\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}\\
&+\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t}\Bigg]\\
&+\left(\Big(g^{jr}g^{lp}g^{sq}+g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{jr}g^{ls}Ric_{rs}\right)\\
&\left(\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}+(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= -2g^{jl}Ric_{js}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)-2g^{jl}Ric_{jr}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+g^{jl}g_{js}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)+g^{jl}g_{jr}\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+2g^{jl}\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}+2g^{jl}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\frac{\partial g_{jr}}{\partial t}\\
&-2g^{jr}g^{ls}g_{js}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t}\\
&-2g^{jr}g^{ls}g_{jr}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+\left(\Big(g^{jr}g^{lp}g^{sq}+g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{jr}g^{ls}Ric_{rs}\right)\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)g_{js}\\
&+\left(\Big(g^{jr}g^{lp}g^{sq}+g^{ls}g^{jp}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{jr}g^{ls}Ric_{rs}\right)(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})g_{jr}\quad .
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= -2g^{jl}Ric_{js}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2g^{jl}Ric_{jr}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \frac{\partial^2}{\partial t^2}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ 2g^{jl}\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)\frac{\partial g_{js}}{\partial t} + 2g^{jl}\frac{\partial g_{jr}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&- 2g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&- 2ng^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \left(\left(g_{js}g^{jr}g^{lp}g^{sq} + g_{js}g^{ls}g^{jp}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g_{js}g^{jr}g^{ls}Ric_{rs}\right)\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&+ \left(\left(g_{jr}g^{jr}g^{lp}g^{sq} + g_{jr}g^{ls}g^{jp}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g_{jr}g^{jr}g^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= -2g^{jl}Ric_{js}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2g^{jl}Ric_{jr}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \frac{\partial^2}{\partial t^2}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ 2g^{jl}\frac{\partial g_{js}}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) + 2g^{jl}\frac{\partial g_{jr}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&- 2g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&- 2ng^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \left(\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ls}Ric_{rs}\right)\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&+ \left(\left(ng^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2ng^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\right).
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= -2g^{jl}Ric_{js}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) - 2g^{jl}Ric_{jr}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \left(\frac{\partial}{\partial x^k}\left(\frac{\partial^2 \Gamma^s_{il}}{\partial t^2}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2}\right)\right) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+2g^{jl}\frac{\partial g_{js}}{\partial t}\left(\frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^s_{il}}{\partial t}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\right)\right) + 2g^{jl}\frac{\partial g_{jr}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&-2g^{ls}\frac{\partial g_{rs}}{\partial t}\left(\frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^s_{il}}{\partial t}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\right)\right) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&-2ng^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \left(\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ls}Ric_{rs}\right)\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
&+ \left(\left(ng^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2ng^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\right).
\end{aligned} \tag{3.13}$$

En mettant en évidence les coefficients des termes $(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks})$, $\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks})$, $\left(\frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^s_{il}}{\partial t}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\right)\right)$ et $\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)$, l'expression (3.13) devient :

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left[\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t} - 2g^{jl}Ric_{js}\right]. \\
&\left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) + \left(\frac{\partial}{\partial x^k}\left(\frac{\partial^2 \Gamma^s_{il}}{\partial t^2}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2}\right)\right) \\
&+ \left(2g^{jl}\frac{\partial g_{js}}{\partial t} - 2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\left(\frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^s_{il}}{\partial t}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\right)\right) \\
&+ \left[\left(ng^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2ng^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} - 2g^{jl}Ric_{jr}\right]. \\
&(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
&+ \left(2g^{jl}\frac{\partial g_{jr}}{\partial t} - 2ng^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\left(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \quad = \quad & \left[\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t} - 2g^{jl}Ric_{js}\right]. \\
& \left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
& + \left[\left(ng^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2ng^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} - 2g^{jl}Ric_{jr}\right]. \\
& (\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
& + \left(2g^{jl}\frac{\partial g_{jr}}{\partial t} - 2ng^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
& + \left(\frac{\partial}{\partial x^k}\left(\frac{\partial^2 \Gamma^s_{il}}{\partial t^2}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2}\right)\right) \\
& + \left(2g^{jl}\frac{\partial g_{js}}{\partial t} - 2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\left(\frac{\partial}{\partial x^k}\left(\frac{\partial \Gamma^s_{il}}{\partial t}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\right)\right).
\end{aligned} \tag{3.14}$$

D'après le **Lemme 3.3** et la **Proposition 3.1**, on a

$$\frac{\partial \Gamma^s_{ik}}{\partial t} = \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) - \frac{1}{2}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right). \tag{3.15}$$

$$\begin{aligned}
\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2} \quad = \quad & \frac{1}{2}\frac{\partial^2 g^{sj}}{\partial t^2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) + \frac{\partial g^{sj}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) \\
& + \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^k}(\frac{\partial^2 g_{ji}}{\partial t^2}) + \frac{\partial}{\partial x^i}(\frac{\partial^2 g_{jk}}{\partial t^2}) - \frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ik}}{\partial t^2})\right) \\
= \quad & \frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk} + g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) \\
& - g^{sp}g^{jq}\frac{\partial g^{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) \\
& + \frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^k}(\frac{\partial^2 g_{ji}}{\partial t^2}) + \frac{\partial}{\partial x^i}(\frac{\partial^2 g_{jk}}{\partial t^2}) - \frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ik}}{\partial t^2})\right).
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2} &= \frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk} + g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) \\
& -g^{sp}g^{jq}\frac{\partial g^{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) \\
& -g^{sj}\left(\frac{\partial}{\partial x^k}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jk}) - \frac{\partial}{\partial x^j}(Ric_{ik})\right).
\end{aligned}$$

Alors l'expression (3.14) , devient :

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} = & \left[\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2g^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t} - 2g^{jl}Ric_{js}\right] \\
& \left(\frac{\partial \Gamma^s_{il}}{\partial x^k} - \frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
& + \left[\left(ng^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} + 2ng^{ls}Ric_{rs} - 2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} - 2g^{jl}Ric_{jr}\right] \\
& (\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
& + \left(2g^{jl}\frac{\partial g_{jr}}{\partial t} - 2ng^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) + \frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \\
& + \frac{\partial}{\partial x^k}\left\{\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk} + g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\right)\right. \\
& \left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right) \\
& - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
& \left. - g^{sj}\left(\frac{\partial}{\partial x^l}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jl}) - \frac{\partial}{\partial x^j}(Ric_{il})\right)\right\} \\
& - \frac{\partial}{\partial x^l}\left\{\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk} + g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\right)\right. \\
& \left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) \\
& - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) \\
& \left. - g^{sj}\left(\frac{\partial}{\partial x^k}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jk}) - \frac{\partial}{\partial x^j}(Ric_{ik})\right)\right\} \\
& + \left(2g^{jl}\frac{\partial g_{js}}{\partial t} - 2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\left\{\frac{\partial}{\partial x^k}\left(\frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})\right.\right.\right. \\
& \left. - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) - \frac{1}{2}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right)\Bigg) \\
& - \frac{\partial}{\partial x^l}\left(\frac{1}{2}g^{sj}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\right. \\
& \left.\left. - \frac{1}{2}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \quad = \quad & \left[\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}-2g^{jl}Ric_{js}\right] \\
& \left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}\right) \\
& +\left[\left(ng^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2ng^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}-2g^{jl}Ric_{jr}\right] \\
& \left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right) \\
& +\left(2g^{jl}\frac{\partial g_{jr}}{\partial t}-2ng^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})+\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right) \\
& +\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right] \\
& \left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right) \\
& +\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right) \\
& \frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right) \\
& -\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
& -g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
& -g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
& -\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right) \\
& -g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right) \\
& -\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right] \\
& \left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right) \\
& -\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right) \\
& \frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right) \\
& +\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right) \\
& +g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\frac{1}{2}g^{sj}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{1}{2}g^{sp}g^{jq}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
&-\frac{1}{2}g^{sp}g^{jq}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
&-\frac{1}{2}g^{sj}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{1}{2}g^{sp}g^{jq}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\\
&+\frac{1}{2}g^{sp}g^{jq}\left(2g^{jl}\frac{\partial g_{js}}{\partial t}-2g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right).
\end{aligned}
$$

En mettant en évidence les termes semblables, on obtient

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} = &\left[\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}-2g^{jl}Ric_{js}\right]\\
&\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\\
&+\left[\left(ng^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2ng^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}-2g^{jl}Ric_{jr}\right]\\
&\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+\left(2g^{jl}\frac{\partial g_{jr}}{\partial t}-2ng^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})+\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right]\\
&\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right]\\
&\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\\
&\left[\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)-\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\right]
\end{aligned}
$$

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$$
\begin{aligned}
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\\
&\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
&+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\right.\\
&\left.-\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\right].
\end{aligned}
\tag{3.16}
$$

Travaillant dans un espace de dimension 3, l'expression (3.16) devient :

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \quad = \quad & \left[\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{js}}{\partial t}-2g^{jl}Ric_{js}\right]\\
&\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)
\end{aligned}
$$

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$$+\left[\left(3g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+6g^{ls}Ric_{rs}-2g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}-2g^{jl}Ric_{jr}\right]$$
$$\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)$$
$$+\left(2g^{jl}\frac{\partial g_{jr}}{\partial t}-6g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})+\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)$$
$$+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right]$$
$$\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)$$
$$-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right]$$
$$\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)$$
$$+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)$$
$$\left[\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)-\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\right]$$
$$-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)$$
$$+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)$$

$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\right.$$
$$\left.-\frac{\partial}{\partial x^l}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\right].$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}+3\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\right)\\
&+g^{ls}g^{rq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}+\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+2g^{ls}Ric_{rs}\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}+3\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\right)\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}+\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+2\left(g^{jl}\frac{\partial g_{jr}}{\partial t}-3g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\\
&+\frac{\partial^2}{\partial t^2}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{il}\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{ik}\\
&+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)(2g_{sj})^2\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}(2g_{sj})\Gamma^s_{il}\\
&+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}(2g_{sj})\Gamma^s_{ik}\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}(2g_{sj})\left(\frac{\partial \Gamma^s_{ik}}{\partial x^k}-\frac{\partial \Gamma^s_{il}}{\partial x^l}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \;=\;& \left(g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}\right)\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}+3\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)\right)\\
&+g^{ls}g^{rq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}\\
&+2\left(g^{jl}\frac{\partial g_{jr}}{\partial t}-3g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big)+\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{il}\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{ik}\\
&+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)(2g_{sj})^2\left(\frac{\partial \Gamma^s_{il}}{\partial x^k}-\frac{\partial \Gamma^s_{ik}}{\partial x^l}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}(2g_{sj})\Gamma^s_{il}\\
&+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}(2g_{sj})\Gamma^s_{ik}\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}(2g_{sj})\left(\frac{\partial\Gamma^s_{ik}}{\partial x^k}-\frac{\partial\Gamma^s_{il}}{\partial x^l}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \;=\; &\left(g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}\right)Ric_{ik}\\
&+2\left(g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+g^{ls}g^{rq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}\\
&+2\left(g^{jl}\frac{\partial g_{jr}}{\partial t}-3g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)+\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{il}\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{ik}\\
&+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)(2g_{sj})^2\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)
\end{aligned}
$$

$$-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}(2g_{sj})\Gamma^s_{il}$$
$$+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}(2g_{sj})\Gamma^s_{ik}$$
$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}(2g_{sj})\left(\frac{\partial\Gamma^s_{ik}}{\partial x^k}-\frac{\partial\Gamma^s_{il}}{\partial x^l}\right).$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}\\
&+2\left(g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+2\left(g^{jl}\frac{\partial g_{jr}}{\partial t}-3g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)+\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks})\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{il}\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{ik}\\
&+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)(2g_{sj})^2\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)
\end{aligned}$$

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$$-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}(2g_{sj})\Gamma^s_{il}$$
$$+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}(2g_{sj})\Gamma^s_{ik}$$
$$-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}(2g_{sj})\left(\frac{\partial\Gamma^s_{ik}}{\partial x^k}-\frac{\partial\Gamma^s_{il}}{\partial x^l}\right)$$

$$\begin{aligned}\frac{\partial^2 Ric_{ik}}{\partial t^2} \quad = \quad & \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right).\end{aligned}$$

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$$-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)$$
$$+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)$$
$$+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)$$
$$-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)$$
$$+B_{ik}\ .$$

Où

$$\begin{aligned}
B_{ik} &= 2\left(g^{lp}g^{sq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2g^{ls}Ric_{rs}\right)\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+2\left(g^{jl}\frac{\partial g_{jr}}{\partial t}-3g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial}{\partial t}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&+\frac{\partial}{\partial x^k}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{il}\\
&-\frac{\partial}{\partial x^l}\left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\right](2g_{sj})\,\Gamma^s_{ik}\\
&+\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qk}+g^{jq}g^{sr}g^{pk}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rk}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)(2g_{sj})^2\left(\frac{\partial\Gamma^s_{il}}{\partial x^k}-\frac{\partial\Gamma^s_{ik}}{\partial x^l}\right)\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^k\partial t}(2g_{sj})\Gamma^s_{il}\\
&+g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial^2 g_{pq}}{\partial x^l\partial t}(2g_{sj})\Gamma^s_{ik}\\
&-g^{sp}g^{jq}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)\frac{\partial g_{pq}}{\partial t}(2g_{sj})\left(\frac{\partial\Gamma^s_{ik}}{\partial x^k}-\frac{\partial\Gamma^s_{il}}{\partial x^l}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\\
&-g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}
\tag{3.17}
$$

De l'autre côté, on a

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right)\right)\\
&= \frac{\partial}{\partial t}\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\Gamma^r_{ls}+\frac{\partial \Gamma^r_{ls}}{\partial t}\Gamma^s_{ik}-\frac{\partial \Gamma^s_{il}}{\partial t}\Gamma^r_{ks}-\frac{\partial \Gamma^r_{ks}}{\partial t}\Gamma^s_{il}\right)\\
&= \frac{\partial^2 \Gamma^s_{ik}}{\partial t^2}\Gamma^r_{ls}+\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}+\frac{\partial^2 \Gamma^r_{ls}}{\partial t^2}\Gamma^s_{ik}+\frac{\partial \Gamma^r_{ls}}{\partial t}\frac{\partial \Gamma^s_{ik}}{\partial t}\\
&\quad -\frac{\partial^2 \Gamma^s_{il}}{\partial t^2}\Gamma^r_{ks}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}-\frac{\partial^2 \Gamma^r_{ks}}{\partial t^2}\Gamma^s_{il}-\frac{\partial \Gamma^r_{ks}}{\partial t}\frac{\partial \Gamma^s_{il}}{\partial t}\quad .
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\left(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\right) &= 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)+\frac{\partial^2 \Gamma^s_{ik}}{\partial t^2}\Gamma^r_{ls}+\frac{\partial^2 \Gamma^r_{ls}}{\partial t^2}\Gamma^s_{ik}\\
&\quad -\frac{\partial^2 \Gamma^s_{il}}{\partial t^2}\Gamma^r_{ks}-\frac{\partial^2 \Gamma^r_{ks}}{\partial t^2}\Gamma^s_{il}\quad .
\end{aligned}
\tag{3.18}
$$

D'après la **Proposition 3.1**, l'expression (3.18) prend la forme suivante

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) \quad = \quad & 2\left(\frac{\partial\Gamma^s_{ik}}{\partial t}\frac{\partial\Gamma^r_{ls}}{\partial t} - \frac{\partial\Gamma^s_{il}}{\partial t}\frac{\partial\Gamma^r_{ks}}{\partial t}\right) + \left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qh} + g^{jq}g^{sr}g^{ph}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}\right.\right.\\
& \left.+2g^{sp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)\\
& -g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
& \left.-g^{sj}\left(\frac{\partial}{\partial x^k}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jk}) - \frac{\partial}{\partial x^j}(Ric_{ik})\right)\right]\\
& \frac{1}{2}g^{rj}\left(\frac{\partial g_{jl}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^l} - \frac{\partial g_{ls}}{\partial x^j}\right) + \left[\frac{1}{2}\left(\left(g^{rs}g^{ph}g^{jq} + g^{rp}g^{js}g^{qh}\right)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right.\right.\\
& \left.+2g^{rp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{jl}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^l} - \frac{\partial g_{ls}}{\partial x^j}\right)\\
& -g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jl}}{\partial t}) + \frac{\partial}{\partial x^l}(\frac{\partial g_{js}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ls}}{\partial t})\right)\\
& \left.-g^{rj}\left(\frac{\partial}{\partial x^s}(Ric_{jl}) + \frac{\partial}{\partial x^l}(Ric_{js}) - \frac{\partial}{\partial x^j}(Ric_{ls})\right)\right]\\
& \frac{1}{2}g^{sj}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) - \left[\frac{1}{2}\left(\left(g^{sp}g^{jr}g^{qh} + g^{jq}g^{sr}g^{ph}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}\right.\right.\\
& \left.+2g^{sp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right)\\
& -g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
& \left.-g^{sj}\left(\frac{\partial}{\partial x^l}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jl}) - \frac{\partial}{\partial x^j}(Ric_{il})\right)\right]\\
& \frac{1}{2}g^{rj}\left(\frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^j}\right) - \left[\frac{1}{2}\left(\left(g^{rs}g^{ph}g^{jq} + g^{rp}g^{js}g^{qh}\right)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right.\right.\\
& \left.+2g^{rp}g^{jq}Ric_{pq}\right)\left(\frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^j}\right)\\
& -g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jk}}{\partial t}) + \frac{\partial}{\partial x^k}(\frac{\partial g_{js}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ks}}{\partial t})\right)\\
& \left.-g^{rj}\left(\frac{\partial}{\partial x^s}(Ric_{jk}) + \frac{\partial}{\partial x^k}(Ric_{js}) - \frac{\partial}{\partial x^j}(Ric_{ks})\right)\right]\\
& \frac{1}{2}g^{sj}\left(\frac{\partial g_{ji}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{il}}{\partial x^j}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\Big(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}\Big) \quad = \quad & 2\left(\frac{\partial\Gamma^s_{ik}}{\partial t}\frac{\partial\Gamma^r_{ls}}{\partial t}-\frac{\partial\Gamma^s_{il}}{\partial t}\frac{\partial\Gamma^r_{ks}}{\partial t}\right)\\
& +\frac{1}{4}g^{rj}\Bigg(\Big(g^{sp}g^{jr}g^{qh}+g^{jq}g^{sr}g^{ph}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\Bigg)\\
& \left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)-\frac{1}{2}g^{rj}g^{sp}g_{jq}\frac{\partial g_{pq}}{\partial t}\\
& \left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\\
& -\frac{1}{2}g^{rj}g^{sj}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}\right.\\
& \left.-\frac{\partial g_{ls}}{\partial x^j}\right)+\frac{1}{4}g^{sj}\Bigg(\Big(g^{rs}g^{ph}g^{jq}+g^{rp}g^{js}g^{qh}\Big)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}+2g^{rp}g^{jq}Ric_{pq}\Bigg)\\
& \left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)-\frac{1}{2}g^{sj}g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\\
& \left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jl}}{\partial t})+\frac{\partial}{\partial x^l}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ls}}{\partial t})\right)\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\\
& -\frac{1}{2}g^{sj}g^{rj}\left(\frac{\partial}{\partial x^s}(Ric_{jl})+\frac{\partial}{\partial x^l}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ls})\right)\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}\right.\\
& \left.-\frac{\partial g_{ik}}{\partial x^j}\right)-\frac{1}{4}g^{rj}\Bigg(\Big(g^{sp}g^{jr}g^{qh}+g^{jq}g^{sr}g^{ph}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\Bigg)\\
& \left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)+\frac{1}{2}g^{rj}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\\
& \left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)\\
& +\frac{1}{2}g^{rj}g^{sj}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}\right.\\
& \left.-\frac{\partial g_{ks}}{\partial x^j}\right)-\frac{1}{4}g^{sj}\Bigg(\Big(g^{rs}g^{ph}g^{jq}+g^{rp}g^{js}g^{qh}\Big)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}+2g^{rp}g^{jq}Ric_{pq}\Bigg)\\
& \left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)+\frac{1}{2}g^{sj}g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\\
& \left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jk}}{\partial t})+\frac{\partial}{\partial x^k}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ks}}{\partial t})\right)\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\\
& +\frac{1}{2}g^{sj}g^{rj}\left(\frac{\partial}{\partial x^s}(Ric_{jk})+\frac{\partial}{\partial x^k}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ks})\right)\\
& \left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right).
\end{aligned}
$$

Et par la mise en évidence des termes semblables, on aura

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}) &= 2\left(\frac{\partial\Gamma^s_{ik}}{\partial t}\frac{\partial\Gamma^r_{ls}}{\partial t}-\frac{\partial\Gamma^s_{il}}{\partial t}\frac{\partial\Gamma^r_{ks}}{\partial t}\right)\\
&+\frac{1}{4}g^{rj}\Bigg(\left(g^{sp}g^{jr}g^{qh}+g^{jq}g^{sr}g^{ph}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\Bigg)\\
&\Bigg[\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\\
&-\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)\Bigg]+\frac{1}{2}g^{rj}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\\
&\Bigg[\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)-\\
&\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\Bigg]\\
&+\frac{1}{2}g^{rj}g^{sj}\Bigg[\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)-\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\Bigg]\\
&+\frac{1}{4}g^{sj}\Bigg(\left(g^{rs}g^{ph}g^{jq}+g^{rp}g^{js}g^{qh}\right)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}+2g^{rp}g^{jq}Ric_{pq}\Bigg)\\
&\Bigg[\left(\frac{\partial g_{jl}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^l}-\frac{\partial g_{ls}}{\partial x^j}\right)\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\\
&-\left(\frac{\partial g_{jk}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^k}-\frac{\partial g_{ks}}{\partial x^j}\right)\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)\Bigg]+\frac{1}{2}g^{sj}g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\\
&\Bigg[\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jk}}{\partial t})+\frac{\partial}{\partial x^k}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ks}}{\partial t})\right)\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)-\\
&\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jl}}{\partial t})+\frac{\partial}{\partial x^l}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ls}}{\partial t})\right)\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\Bigg]\\
&+\frac{1}{2}g^{rj}g^{sj}\Bigg[\left(\frac{\partial}{\partial x^s}(Ric_{jk})+\frac{\partial}{\partial x^k}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ks})\right)\\
&\left(\frac{\partial g_{ji}}{\partial x^l}+\frac{\partial g_{jl}}{\partial x^i}-\frac{\partial g_{il}}{\partial x^j}\right)-\left(\frac{\partial}{\partial x^s}(Ric_{jl})+\frac{\partial}{\partial x^l}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ls})\right)\\
&\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls}-\Gamma^s_{il}\Gamma^r_{ks}) \quad = \quad & 2\left(\frac{\partial\Gamma^s_{ik}}{\partial t}\frac{\partial\Gamma^r_{ls}}{\partial t}-\frac{\partial\Gamma^s_{il}}{\partial t}\frac{\partial\Gamma^r_{ks}}{\partial t}\right)\\
& +\frac{1}{4}g^{rj}\left(\left(g^{sp}g^{jr}g^{qh}+g^{jq}g^{sr}g^{ph}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t}+2g^{sp}g^{jq}Ric_{pq}\right)\\
& (4g_{pj}g_{qj})\left(\Gamma^p_{ik}\Gamma^q_{ls}-\Gamma^p_{ik}\Gamma^q_{ks}\right)\\
& +\frac{1}{2}g^{rj}g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left[\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)(2g_{pj})\,\Gamma^p_{ks}-\right.\\
& \left.\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)(2g_{pj})\,\Gamma^p_{ls}\right]\\
& +\frac{1}{2}g^{rj}g^{sj}\left[\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)(2g_{pj})\,\Gamma^p_{ks}-\right.\\
& \left.\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)(2g_{pj})\,\Gamma^p_{ls}\right]\\
& +\frac{1}{4}g^{sj}\left(\left(g^{rs}g^{ph}g^{jq}+g^{rp}g^{js}g^{qh}\right)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t}+2g^{rp}g^{jq}Ric_{pq}\right)\\
& (4g_{pj}g_{qj})\left(\Gamma^p_{ls}\Gamma^q_{ik}-\Gamma^p_{il}\Gamma^q_{ks}\right)\\
& +\frac{1}{2}g^{sj}g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\left[\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jk}}{\partial t})+\frac{\partial}{\partial x^k}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ks}}{\partial t})\right)(2g_{pj})\,\Gamma^p_{il}-\right.\\
& \left.\left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jl}}{\partial t})+\frac{\partial}{\partial x^l}(\frac{\partial g_{js}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ls}}{\partial t})\right)(2g_{pj})\,\Gamma^p_{ik}\right]\\
& +\frac{1}{2}g^{rj}g^{sj}\left[\left(\frac{\partial}{\partial x^s}(Ric_{jk})+\frac{\partial}{\partial x^k}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ks})\right)(2g_{pj})\,\Gamma^p_{il}-\right.\\
& \left.\left(\frac{\partial}{\partial x^s}(Ric_{jl})+\frac{\partial}{\partial x^l}(Ric_{js})-\frac{\partial}{\partial x^j}(Ric_{ls})\right)(2g_{pj})\,\Gamma^p_{ik}\right].
\end{aligned}
$$

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$$\begin{aligned}
\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) &= 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right) \\
&+ g_{pj}\Bigg(\left(g^{sp}g^{jr}g^{qh} + g^{jq}g^{sr}g^{ph}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rh}}{\partial t} + 2g^{sp}g^{jq}Ric_{pq}\Bigg) \\
&\left(\Gamma^p_{ik}\Gamma^q_{ls} - \Gamma^p_{ik}\Gamma^q_{ks}\right) + g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\Bigg[\Bigg(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) \\
&- \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\Bigg)\Gamma^p_{ks} - \left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\Gamma^p_{ls}\Bigg] \\
&+ g^{sj}\Bigg[\left(\frac{\partial}{\partial x^l}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jl}) - \frac{\partial}{\partial x^j}(Ric_{il})\right)\Gamma^p_{ks} - \\
&\left(\frac{\partial}{\partial x^k}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jk}) - \frac{\partial}{\partial x^j}(Ric_{ik})\right)\Gamma^p_{ls}\Bigg] \\
&+ g^{pj}\Bigg(\left(g^{rs}g^{ph}g^{jq} + g^{rp}g^{js}g^{qh}\right)\frac{\partial g_{sh}}{\partial t}\frac{\partial g_{pq}}{\partial t} + 2g^{rp}g^{jq}Ric_{pq}\Bigg) \\
&(\Gamma^p_{ls}\Gamma^q_{ik} - \Gamma^p_{il}\Gamma^q_{ks}) + g^{rp}g^{jq}\frac{\partial g_{pq}}{\partial t}\Bigg[\Big(\frac{\partial}{\partial x^s}(\frac{\partial g_{jk}}{\partial t}) + \frac{\partial}{\partial x^k}(\frac{\partial g_{js}}{\partial t}) \\
&- \frac{\partial}{\partial x^j}(\frac{\partial g_{ks}}{\partial t})\Big)\Gamma^p_{il} - \left(\frac{\partial}{\partial x^s}(\frac{\partial g_{jl}}{\partial t}) + \frac{\partial}{\partial x^l}(\frac{\partial g_{js}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{ls}}{\partial t})\right)\Gamma^p_{ik}\Bigg] \\
&+ g^{sj}\Bigg[\left(\frac{\partial}{\partial x^s}(Ric_{jk}) + \frac{\partial}{\partial x^k}(Ric_{js}) - \frac{\partial}{\partial x^j}(Ric_{ks})\right)\Gamma^p_{il} - \\
&\left(\frac{\partial}{\partial x^s}(Ric_{jl}) + \frac{\partial}{\partial x^l}(Ric_{js}) - \frac{\partial}{\partial x^j}(Ric_{ls})\right)\Gamma^p_{ik}\Bigg].
\end{aligned}$$

Finalement, on obtient

$$\frac{\partial^2}{\partial t^2}(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}) = 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right). \tag{3.19}$$

Par ailleurs l'expression (3.17) devient

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik} + 2g^{ls}Ric_{rs}Ric_{ik} \\
&- 2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Ric_{ik} + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right) \\
&- g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^k \partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
&- \frac{\partial}{\partial x^k}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t}) + \frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right) \\
&- \frac{\partial g^{sj}}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji}) + \frac{\partial}{\partial x^i}(Ric_{jl}) - \frac{\partial}{\partial x^j}(Ric_{il})\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial}{\partial x^l}\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\frac{\partial g^{sj}}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+g^{sp}g^{jq}\frac{\partial^2 g_{pq}}{\partial x^l\partial t}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sj}\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jl})-\frac{\partial}{\partial x^j}(Ric_{il})\right)\\
&+g^{sj}\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(Ric_{ji})+\frac{\partial}{\partial x^i}(Ric_{jk})-\frac{\partial}{\partial x^j}(Ric_{ik})\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\\
&\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^l}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\\
&\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^k}(\frac{\partial g_{ji}}{\partial t})+\frac{\partial}{\partial x^i}(\frac{\partial g_{jk}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}
\tag{3.20}
$$

Avec $\dfrac{\partial}{\partial x^i}=\nabla_i$, on obtient l'expression suivante :

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$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik} + 2g^{ls}Ric_{rs}Ric_{ik} \\
&\quad -2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Ric_{ik} + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right) \\
&\quad -\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&\quad -\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{li}\Big) \\
&\quad +\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&\quad +\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\
&\quad -g^{sp}g^{jq}\nabla_k\left(\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&\quad +g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&\quad -g^{sj}\nabla_k\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{il}\Big) \\
&\quad +g^{sj}\nabla_l\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\
&\quad +\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t} - g^{ls}\frac{\partial g_{rs}}{\partial t}\right) - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)
\end{aligned}$$

$$\begin{aligned}
&\nabla_k\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t} - g^{ls}\frac{\partial g_{rs}}{\partial t}\right) - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right) \\
&\nabla_l\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\\
&-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big)\\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big)\\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&-g^{sj}\left(\nabla_k\nabla_l Ric_{ji}+\nabla_k\nabla_i Ric_{jl}-\nabla_k\nabla_j Ric_{il}\right)\\
&+g^{sj}\left(\nabla_l\nabla_k Ric_{ji}+\nabla_l\nabla_i Ric_{jk}-\nabla_l\nabla_j Ric_{ik}\right)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\\
&\left(\nabla_k\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&-\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\\
&\left(\nabla_l\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\\
&-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)
\end{aligned}$$

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$$
\begin{aligned}
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big)\\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big)\\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+g^{sj}\Big(\nabla_l\nabla_k Ric_{ji}+\nabla_l\nabla_i Ric_{jk}-\nabla_l\nabla_j Ric_{ik}\\
&-\nabla_k\nabla_l Ric_{ji}-\nabla_k\nabla_i Ric_{jl}+\nabla_k\nabla_j Ric_{il}\Big)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})\right.\\
&\left.-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})-\nabla_l\nabla_k(\frac{\partial g_{ji}}{\partial t})-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})+\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} \;=\;& \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik}\\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\\
&-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big)\\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big)\\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+g^{sj}\left(\nabla_l\nabla_i Ric_{jk}-\nabla_l\nabla_j Ric_{ik}-\nabla_k\nabla_i Ric_{jl}+\nabla_k\nabla_j Ric_{il}\right)(s\leftrightarrow l)\\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right.\\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})+\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}+2g^{ls}Ric_{rs}Ric_{ik} \\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik}+2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right) \\
&-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big) \\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big) \\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+g^{lj}\left(\nabla_l\nabla_i Ric_{jk}-\nabla_l\nabla_j Ric_{ik}-\nabla_k\nabla_i Ric_{jl}+\nabla_k\nabla_j Ric_{il}\right) \\
&+\left(g^{sj}\left(g^{jl}\frac{\partial g_{js}}{\partial t}-g^{ls}\frac{\partial g_{rs}}{\partial t}\right)-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right. \\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})+\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \Delta Ric_{ik}+\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik}-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)Ric_{ik} \\
&+2g^{ls}Ric_{rs}Ric_{ik}-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big) \\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big) \\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+g^{lj}\left(\nabla_l\nabla_i Ric_{jk}-\nabla_k\nabla_i Ric_{jl}+\nabla_k\nabla_j Ric_{il}\right) \\
&+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t}-g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t}-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right. \\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})+\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)+2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t}-\frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right).
\end{aligned}$$

(3.21)

D'où la proposition suivante :

Proposition 3.3. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t,x,y)\Big)$. Alors sous le flux géométrique hyperbolique d'un espace berwaldien, le tenseur de Ricci satisfait à l'équation d'évolution suivante :*

$$\begin{aligned}\frac{\partial^2 Ric_{ik}}{\partial t^2} &= \Delta Ric_{ik} + \left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik} - 2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Ric_{ik} \\ &\quad +2g^{ls}Ric_{rs}Ric_{ik} + \mathcal{N}_{ik} \qquad (3.22)\end{aligned}$$

où

$$\begin{aligned}\mathcal{N}_{ik} &= -\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\ &\quad -\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{li}\Big) \\ &\quad +\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\ &\quad +\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\ &\quad -g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\ &\quad +g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\ &\quad +g^{lj}\left(\nabla_l\nabla_i Ric_{jk} - \nabla_k\nabla_i Ric_{jl} + \nabla_k\nabla_j Ric_{il}\right) \\ &\quad +\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t} - g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t} - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right. \\ &\quad \left. -\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t}) + \nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\end{aligned}$$

(3.23)

avec $\Delta = g^{jl}\nabla_j\nabla_l$.

Corollaire 3.1. *Si M est un espace d'Einstein, i.e $Ric_{ij} = cg_{ij}$, avec c étant une constante, alors*

$$\begin{aligned}
\frac{\partial^2 Ric_{ik}}{\partial t^2} &= c\,\Delta g_{ik} + c\,g_{ik}\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} - 2\,c\,g_{ik}\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+c\right)\\
&+2\,c^2\,g_{ik} - \nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&-c\,\nabla_k(g^{sj})\Big(\nabla_l g_{ji} + \nabla_i g_{jl} - \nabla_j g_{li}\Big)
\end{aligned}$$

$$\begin{aligned}
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+c\nabla_l(g^{sj})\Big(\nabla_k g_{ji} + \nabla_i g_{jk} - \nabla_j g_{ik}\Big)\\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+cg^{lj}\left(\nabla_l\nabla_i g_{jk} - \nabla_k\nabla_i g_{jl} + \nabla_k\nabla_j g_{il}\right)\\
&+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t} - g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t} - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right.\\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t}) + \nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right).
\end{aligned} \tag{3.24}$$

3.4 Equations d'évolution hyperbolique de la courbure scalaire berwaldienne

En respectant à la courbure scalaire berwaldienne , $Scal = g^{ik}Ric_{ik}$, on a les équations d'évolution suivantes

$$\frac{\partial Scal}{\partial t} = \frac{\partial g^{ik}}{\partial t}Ric_{ik} + g^{ik}\frac{\partial Ric_{ik}}{\partial t} \tag{3.25}$$

$$\frac{\partial^2 Scal}{\partial t^2} = \frac{\partial g^{ik}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} + \frac{\partial^2 g^{ik}}{\partial t^2}Ric_{ik} + \frac{\partial g^{ik}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} + g^{ik}\frac{\partial^2 Ric_{ik}}{\partial t^2} \tag{3.26}$$

$$= 2\frac{\partial g^{ik}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} + \frac{\partial^2 g^{ik}}{\partial t^2}Ric_{ik} + g^{ik}\frac{\partial^2 Ric_{ik}}{\partial t^2}. \tag{3.27}$$

Sachant que l'élément g^{ik} du tenseur fondamental satisfait à l'évolution suivante :

$$\frac{\partial g^{ik}}{\partial t} = -g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t} \tag{3.28}$$

et d'après le **Lemme 3.2**, alors

$$\begin{aligned} \frac{\partial^2 g^{ik}}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial g^{ik}}{\partial t}\right) \\ &= \frac{\partial}{\partial t}\left(-g^{ip}g^{kq}\right)\frac{\partial g_{pq}}{\partial t} - g^{ip}g^{kq}\frac{\partial^2 g_{pq}}{\partial t^2} \\ &= \left(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\right)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t} - g^{ip}g^{kq}\frac{\partial^2 g_{pq}}{\partial t^2} \quad . \end{aligned}$$

Par la **Définition3.1**, on obtient

$$\frac{\partial^2 g^{ik}}{\partial t^2} = \left(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\right)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t} + 2g^{ip}g^{kq}Ric_{pq} \quad . \tag{3.29}$$

En faisant le remplacement de (3.28) et (3.29) dans (3.27) on obtient

$$\begin{aligned} \frac{\partial^2 Scal}{\partial t^2} &= -2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} + \left(\left(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\right)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t} + 2g^{ip}g^{kq}Ric_{pq}\right)Ric_{ik} \\ &+ g^{ik}\Bigg[\Delta Ric_{ik} + \left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik} \\ &-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Ric_{ik} \\ &+2g^{ls}Ric_{rs}Ric_{ik} - \nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\ &-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{li}\Big) \\ &+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\ &+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\ &-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\ &+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\ &+g^{lj}\left(\nabla_l\nabla_i Ric_{jk} - \nabla_k\nabla_i Ric_{jl} + \nabla_k\nabla_j Ric_{il}\right) \\ &+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t} - g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t} - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right. \\ &\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t}) + \nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\Bigg] \end{aligned}$$

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(3.30)

$$\begin{aligned}
\frac{\partial^2 Scal}{\partial t^2} &= -2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} + \left(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\right)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t}Ric_{ik} \\
&+2g^{ip}g^{kq}Ric_{pq}Ric_{ik} + g^{ik}\Delta Ric_{ik} + g^{ik}\left(g^{lp}g^{sq} + g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Ric_{ik} \\
&-2g^{ik}\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Ric_{ik} + 2g^{ik}g^{ls}Ric_{rs}Ric_{ik} \\
&+g^{ik}\Big[-\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{li}\Big)
\end{aligned}$$

$$\begin{aligned}
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+g^{lj}\left(\nabla_l\nabla_i Ric_{jk} - \nabla_k\nabla_i Ric_{jl} + \nabla_k\nabla_j Ric_{il}\right) \\
&+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t} - g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t} - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\Big(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t}) \\
&-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t}) + \nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\Big) + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\Big]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Scal}{\partial t^2} &= \Delta Scal + 2|Ric|^2 + \Big(g^{lp}g^{sq} + g^{ls}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Scal + 2g^{ls}Ric_{rs}Scal \\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Scal - 2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} \\
&+ \Big(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\Big)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t}Ric_{ik} \\
&+g^{ik}\Bigg[-\nabla_k\Big(g^{sp}g^{jq}\Big)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji} + \nabla_i Ric_{jl} - \nabla_j Ric_{li}\Big) \\
&+\nabla_l\Big(g^{sp}g^{jq}\Big)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji} + \nabla_i Ric_{jk} - \nabla_j Ric_{ik}\Big) \\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_j(\frac{\partial g_{il}}{\partial t})\right) \\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t}) + \nabla_i(\frac{\partial g_{jk}}{\partial t}) - \nabla_j(\frac{\partial g_{ik}}{\partial t})\right) \\
&+g^{lj}\left(\nabla_l\nabla_i Ric_{jk} - \nabla_k\nabla_i Ric_{jl} + \nabla_k\nabla_j Ric_{il}\right) \\
&+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t} - g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t} - g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t}) - \nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right. \\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t}) + \nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right) + 2\left(\frac{\partial \Gamma^s_{ik}}{\partial t}\frac{\partial \Gamma^r_{ls}}{\partial t} - \frac{\partial \Gamma^s_{il}}{\partial t}\frac{\partial \Gamma^r_{ks}}{\partial t}\right)\Bigg].
\end{aligned}$$

D'où la proposition suivante :

Proposition 3.4. *Soit (M, F_t) un espace berwaldien de tenseur fondamental $g = \Big(g_{ij}(t,x,y)\Big)$. Alors sous le flux géométrique hyperbolique d'un espace berwaldien, le tenseur de la courbure scalaire* ***Scal*** *satisfait à l'équation d'évolution suivante :*

$$\begin{aligned}
\frac{\partial^2 Scal}{\partial t^2} &= \Delta Scal + 2|Ric|^2 + \Big(g^{lp}g^{sq} + g^{ls}g^{rq}\Big)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}Scal + 2g^{ls}Ric_{rs}Scal \\
&-2\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t} + g^{jl}Ric_{jr}\right)Scal - 2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial Ric_{ik}}{\partial t} \\
&+ \Big(g^{ip}g^{kr}g^{qs} + g^{kq}g^{ir}g^{ps}\Big)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t}Ric_{ik} + g^{ik}\mathcal{N}_{ik} \quad .
\end{aligned} \tag{3.31}$$

Où

$$\begin{aligned}
\mathcal{N}_{ik} &= -\nabla_k\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&-\nabla_k(g^{sj})\Big(\nabla_l Ric_{ji}+\nabla_i Ric_{jl}-\nabla_j Ric_{li}\Big)\\
&+\nabla_l\left(g^{sp}g^{jq}\right)\frac{\partial g_{pq}}{\partial t}\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+\nabla_l(g^{sj})\Big(\nabla_k Ric_{ji}+\nabla_i Ric_{jk}-\nabla_j Ric_{ik}\Big)\\
&-g^{sp}g^{jq}\nabla_k(\frac{\partial g_{pq}}{\partial t})\left(\nabla_l(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_j(\frac{\partial g_{il}}{\partial t})\right)\\
&+g^{sp}g^{jq}\nabla_l(\frac{\partial g_{pq}}{\partial t})\left(\nabla_k(\frac{\partial g_{ji}}{\partial t})+\nabla_i(\frac{\partial g_{jk}}{\partial t})-\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)\\
&+g^{lj}\left(\nabla_l\nabla_i Ric_{jk}-\nabla_k\nabla_i Ric_{jl}+\nabla_k\nabla_j Ric_{il}\right)\\
&+\left(g^{sj}g^{jl}\frac{\partial g_{js}}{\partial t}-g^{sj}g^{ls}\frac{\partial g_{rs}}{\partial t}-g^{sp}g^{jq}\frac{\partial g_{pq}}{\partial t}\right)\left(\nabla_k\nabla_i(\frac{\partial g_{jl}}{\partial t})-\nabla_k\nabla_j(\frac{\partial g_{il}}{\partial t})\right.\\
&\left.-\nabla_l\nabla_i(\frac{\partial g_{jk}}{\partial t})+\nabla_l\nabla_j(\frac{\partial g_{ik}}{\partial t})\right)+2\left(\frac{\partial\Gamma^s_{ik}}{\partial t}\frac{\partial\Gamma^r_{ls}}{\partial t}-\frac{\partial\Gamma^s_{il}}{\partial t}\frac{\partial\Gamma^r_{ks}}{\partial t}\right)
\end{aligned}$$

avec $|Ric|^2 = g^{ip}g^{kq}Ric_{pq}Ric_{ik}$.

Corollaire 3.2. *Si M est un espace d'Einstein $Scal = f(t)$, alors*

$$\begin{aligned}
\frac{\partial^2 Scal}{\partial t^2} &= \Delta f(t)+2|Ric|^2+f(t)\left(g^{lp}g^{sq}+g^{ls}g^{rq}\right)\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}+2f(t)g^{ls}Ric_{rs}\\
&-2f(t)\left(g^{jr}g^{ls}\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{jr}}{\partial t}+g^{jl}Ric_{jr}\right)-2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial Ric_{ik}}{\partial t}\\
&+\left(g^{ip}g^{kr}g^{qs}+g^{kq}g^{ir}g^{ps}\right)\frac{\partial g_{rs}}{\partial t}\frac{\partial g_{pq}}{\partial t}Ric_{ik}+g^{ik}\mathcal{N}_{ik}\quad .
\end{aligned} \tag{3.32}$$

L'évolution de la courbure scalaire par la **Proposition 3.4** fournit une illustration du fait que le flux géométrique hyperbolique préfère la courbure positive.

Dans ce cas, les deux composantes $\Delta Scal$ et $2|Ric|^2$ peuvent être interpretées de la manière suivante :

Le terme dissipatif $\Delta Scal$ reflète le fait qu'un point dans M avec une courbure moyenne élevée que ses voisins aura tendance à inverser à la moyenne.
Le terme non linéaire $2|Ric|^2$ reflète le fait que si l'on est dans une région de courbure positive (par exemple dans une région se comportant comme une sphère), alors la métrique se contractera, augmentant ainsi la courbure pour être encore plus positive.

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Inversement, si l'on se trouve dans une région à courbure négative (comme une région se comportant comme une selle), alors la métrique va se développer, affaiblissant ainsi la négativité de courbure.
Dans les deux cas, la courbure tend vers le haut ce qui est cohérent avec la non-négativité de $2|Ric|^2$.

Les équations d'évolution hyperbolique établies sont d'une façon générale résumées dans la proposition suivante

Proposition 3.5. *Soit* (M, F_t) *un espace berwaldien de tenseur fondamental* $g = \Big(g_{ij}(t, x, y)\Big)$. *Alors sous le flux géométrique hyperbolique, les tenseurs de courbures satisfont les équations d'évolution*

$$\frac{\partial^2 Ric_{ik}}{\partial t^2} = \Delta Ric_{ik} + (termes\ inférieurs) \tag{3.33}$$

$$\frac{\partial^2 Scal}{\partial t^2} = \Delta Scal + (termes\ inférieurs) \tag{3.34}$$

où Δ est le Laplacien par rapport à l'évolution de la métrique, les termes d'ordre inférieurs contiennent uniquement des termes d'ordre inférieur des dérivées des courbures.
Les équations (3.33) et (3.34) montrent que les courbures possèdent une propriété ondulatoire intéressante.

Conclusion

Dans ce mémoire, on a établi les équations d'évolution hyperbolique des tenseurs des courbures berwaldiennes le long du flux géométrique hyperbolique :

$$\frac{\partial^2}{\partial t^2} g(t) = -2Ric_{F(t)}.$$

Ces équations d'évolution hyperbolique établies sont celles trouvées dans la **Proposition 3.3** et dans la **Proposition 3.4** .

Ce travail pourra constituer un outil de base efficace au lecteur qui voudrait traiter la non-négativité des courbures berwaldiennes et la solution satisfaisant le flux géométrique hyperbolique dans les espaces berwaldiens.

Maintenant que nous avons les formes locales des équations d'évolution hyperbolique le long du flux géométrique hyperbolique, donc à l'avenir nous nous intéressons aux problèmes fondamentaux suivants :

1. L'existence à long terme, la formation de singularités, ainsi que les applications physiques et géométriques.
2. Si nous voulons étudier les propriétés locales du flux géométrique hyperbolique, alors c'est très important de trouver les conditions préservées sous l'évolution. Comment développer de telles techniques ?

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