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q-Nonuniform difference classicalanalysis

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q-Nonuniform difference classical analysis

Jean Paul Nuwacu

PhD Thesis

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To my late father Damas Ndabarushimana [1960-1996] and my late grand brother Didier Niyonkuru [1979-1993].

A winner is a dreamer who never gives up

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Abstract

This thesis is concerned in q-nonuniform difference classical analysis that is classical analysis on q-nonuniform lattices.

The first two Chapters treat the standard fondements of this analysis that is the q-nonuniform difference and integral calculus on one side and the q-nonuniform difference equations on the other side.

The third Chapter sumirizes one of the advenced q-nonuniform calculus that is the q-nonuniform difference control theory. This is one of the proof that the q-nonuniform difference analysis has a coherent theory.

The Chapters four and five are concerned in the main contributions of this thesis. The fourth deals with the q-nonuniform orthogonal polynomials and show how to obtain the multiple polynomial of Askey-Wilson. We first show how one can obtain Al-Salam-Chihara polynomials, continuous dual q-Hahn polynomials, and Askey-Wilson polynomials from the little q-Laguerre and the little q-Jacobi polynomials by using special transformations as showed in the figure 4.1. This

procedure is then extended to obtain multiple Askey-Wilson, multiple continuous dual q-Hahn, and multiple Al-Salam-Chihara polynomials from the multiple little q-Laguerre and the multiple little q-Jacobi polynomials.

In the fifth, we prove the essential of classical integral inequalities that is the q-nonuniform integral inequalities of Hölder, Cauchy-Shwartz, and Minkowski based on the techniques used in classical analysis, q-nonuniform integral inequalities of Grönwall and Bernoulli based on the linear q-nonuniform difference equations of first order and the corresponding Lagrange resolution method and the q-nonuniforn integral inequality of Lyapunov based on q-nonuniform Sturm-Liouville equation.

Finally, in the Chapter six, we judged better to expose one of the recent result of q-nonuniform difference analysis that is the q-nonuniform Laplace transform. It is a generalization of the Laplace transfrom, Z-transform and q-Laplace transfrom applied respectively in differential analysis, difference analysis and q-difference analysis.

Résumé

Cette thèse s'intéresse à l'analyse classique aux différences q-non uniformes qui est une analyse classique sur le réseau q-non uniforme.

Les deux premiers chapitres traitent les fondements standards de cette analyse, à savoir le calcul différentiel et intégral sur le réseau q-non uniforme d'un côté et les équations différentielles sur le réseau q-non uniforme de l'autre côté.

Le troisième chapitre résume l'un des calculs avancés sur le réseau q-non uniforme qui est la théorie de contrôle sur le réseau q-non uniforme. C'est l'une des preuves que l'analyse mathématique sur le réseau q-non uniforme admet une théorie cohérente.

Les chapitres quatre et cinq sont concernés par la contribution principale de cette thèse. Le quatrième chapitre traite des polynômes orthogonaux sur le réseau q-non uniforme et montre comment obtenir les polynômes d'Askey-Wilson multiples. Nous montrons d'abord comment obtenir les polynômes d'Al-Salam-Chihara, de q-Hahn duals continus et les polynômes d'Askey-Wilson à partir des polynômes dits petits q-Laguerre et petits q-Jacobi, en utilisant des transformations spéciales comme illustré sur la figure 4.1. Cette procédure est

ensuite étendue pour obtenir les polynômes d'Askey-Wilson multiples, ceux dits q-Hahn duals continus multiples et Al-Salam-Chihara multiples à partir des polynômes dits petits q-Laguerre multiples et les polynmes dits petits q-Jacobi multiples.

Dans le cinquième chapitre, nous prouvons l'essentiel des inégalités intégrales classiques, à savoir les inégalités intégrales sur le réseau q-non uniforme de Hölder, Cauchy-Shwartz et Minkowski basées sur les techniques utilisées en analyse classique, les inégalités intégrales sur le réseau q-non uniforme de Grönwall et de Bernoulli basées sur les équations aux différences linéaires q-non uniformes de premier ordre et la méthode de résolution de Lagrange correspondante, et en fin l'inégalité intégrale sur le réseau q-non uniforme de Lyapunov basée sur l' équation q-non uniforme de Sturm-Liouville.

Dans le dernier chapitre, nous avons jugé préférable d'exposer l'un des résultats récents de l'analyse aux différences sur le réseau q-non uniforme, à savoir la transformée q-non uniforme de Laplace. Il s'agit d'une généralisation de la transformée de Laplace, de la transformée en Z et de la q-transformée de Laplace en analyse différentielle, aux différences et aux q-diffrences réspectivement.

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Introduction

Calculus on lattices appeared as part of the bubbling development of the numerical calculus accompanying the wellknown numerical era beginning with the second part of the 20th century.

The concept of calculus on lattices was initiated by Alphonse P. Magnus in [43, 44] while generalizing orthogonality measure for polynomials. A set of divided difference derivatives on lattices transforming a polynomial of degree n into a polynomial of degree $n-1$ was there naturally derived. This set includes: the standard differential derivative, the difference derivative, the q-derivative or Jackson derivative and the Askey-Wilson derivative, seen as the divided difference derivative on constant, uniform, q-uniform and q-nonuniform lattices respectively.

We refer to the corresponding calculus and equations as the differential (usual), difference, q-uniform difference and q-nonuniform difference calculus and equations respectively.

The differential and difference calculus are already proud of their classificality. At the beginning, q-calculus (or q-uniform difference calculus) appeared as a new field in analysis for the investigation of calculus without limits. F. H. Jackson introduced in [33] the q-calculus in the early twentieth century, but the history of q-calculus (and q-hypergeometric functions) dates back to the eighteenth century.

q-Calculus can in fact be taken as far back as Leonhard Euler (1707-1783), who first introduced the q in [28] in the tracks of Newton's infinite series.

Recently it arose interest due to high demand of mathematics that models quantum computing. Indeed, q-calculus has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences. The book by Kac and Cheung [34] covers many of the fundamental aspects of q-calculus or quantum calculus.

From the beginning of eighties, Alphonse P. Magnus rediscovered and extended the q-calculus and his special cases, in his pionnering work [43]. There, he considered the most general divided difference derivative [43, 44],

$$\mathcal{D}f(t(s)) = \frac{f(t(s+\frac{1}{2}))-f(t(s-\frac{1}{2}))}{t(s+\frac{1}{2})-t(s-\frac{1}{2})}, \tag{1}$$

admitting the property that if $f(t) = P_n(t(s))$ is a polynomial of degree n in $t(s)$, then $\mathcal{D}f(t(s)) = \tilde{P}_{n-1}(t(s))$ is a polynomial in $t(s)$ of degree $n-1$. Here $t(s)$ is a complex-valued discrete variable function satisfying the relation

$$F\Big(t(s), t(s-\frac{1}{2})\Big) = F\Big(t(s), t(s+\frac{1}{2})\Big) = 0, \quad s \in \mathbb{Z}_+, \tag{2}$$

where F is a two variables quadratic polynomial

$$F(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f \tag{3}$$

with $a,\ b,\ c,\ d,\ e,\ f \in \mathbb{C}$.

From (2) and (3) it follows that

$$t(s+\frac{1}{2}) = P(t(s)) + \sqrt{Q(t(s))},\ \ t(s-\frac{1}{2}) = P(t(s)) - \sqrt{Q(t(s))}, \quad (4)$$

where $P(t(s))$ and $Q(t(s))$ are polynomials of degree at most 1 and 2 respectively.

From (4) one derives the following most important canonical forms for $t(s)$ in order of increasing complexity:

$$t(s) = t(0); \quad (5)$$

$$t(s) = s; \quad (6)$$

$$t(s) = q^s; \quad (7)$$

$$t(s) = \frac{q^s + q^{-s}}{2}; q \in \mathbb{C}, s \in \mathbb{Z}. \quad (8)$$

When the function $t(s)$ is given by (5)-(7), the divided difference derivative (1) leads to the ordinary differential derivative $Df(t) = \frac{d}{dt}f(t)$, finite difference derivative

$$\Delta f(s) = f(s+1) - f(s) = (e^{\frac{d}{ds}} - 1)f(s) \quad (9)$$

and q-difference derivative (or Jackson derivative [33])

$$\mathcal{D}_q f(t) = \frac{f(qt) - f(t)}{qt - t} = \frac{q^{\frac{d}{dt}} - 1}{qt - t} f(t) \quad (10)$$

respectively. When $t(s) = x(q^s)$ is given by (8), the corresponding derivative is usually referred to as the *Askey-Wilson* first order divided difference operator [9] that one can write:

$$\mathcal{D}f(x(z)) = \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)}, \quad (11)$$

where $x(z) = \frac{z+z^{-1}}{2}$ is the well known *Joukowski* transformation and $z = q^s$.

The calculus related to the differential derivative, the continuous or differential calculus, is clearly wellknown. The one related to the derivatives (9),(10), (11) (difference, q-difference and q-nonuniform difference respectively) is referred to as the *discrete calculus*. Its interest is two folds: On the one hand, it generalizes the continuous calculus, and on the other hand, it uses a discrete variable.

We will note also that (11) is at our best knowledge, the most general known divided difference derivative having the property of sending a polynomial of degree n in a polynomial of degree $n-1$.

This thesis is concerned in the q-nonuniform difference calculus, that is the calculus generated by the derivative in (11). Our objective is to expose, to introduce or to develop the basic cencepts of q-nonuniform difference analysis or analysis on q-nonuniform lattices, which is currently in its infancy.
In this thesis, we mainly review the foundations already laid and then continue with our own contributions in the area. We begin with the standard preliminaries such as the q-nonuniform difference and integral calculus in Chapter 1, the q-nonuniform difference equations in Chapter 2, continue with the q-nonuniform difference control theory in Chapter 3.

In Chapter 4, we present our own contributions in an important branch of q-nonuniform difference analysis which is the q-nonuniform

difference orthogonal polynomials theory. Here we first show in Section 4.2.2 how some families of basic hypergeometric polynomials are related by linear transformation. This transformation is a q-analogue of the Fourier–Jacobi transform that maps Jacobi polynomials to Wilson polynomials [40].

Next, considering three families of basic hypergeometric polynomials: the Al-Salam–Chihara polynomials, the continuous dual q-Hahn polynomials and the Askey–Wilson polynomials, we show how they can be obtained by a linear transformation from the little q-Laguerre and the little q-Jacobi polynomials. We then extend this procedure to multiple orthogonal polynomials in Section 4.2.3.

Finally, we recall multiple little q-Laguerre polynomials in Section 4.2.3.1 and multiple little q-Jacobi polynomials in Section 4.2.3.2 and then apply the linear transformations to obtain multiple Al-Salam–Chihara polynomials Section 4.2.3.3, multiple continuous dual q-Hahn polynomials Section 4.2.3.4 and finally multiple Askey-Wilson polynomials Section 4.2.3.5.

In Chapter 5, we give our own contributions in another important area of q-nonuniform difference analysis which is the q-nonuniform difference version of the classical integral inequalities. Here, using the techniques used in classical analysis, we establish the q-nonuniform difference versions of the wellknown in differential calculus, integral inequalities of Hölder, Cauchy-Shwartz, and Minkowski (Section 5.1 and Section 5.2). Then using the linear q-nonuniform difference equations of first order and the corresponding Lagrange resolution

method, we establish the q-nonuniform difference versions of the integral inequalities of Grönwall and Bernoulli (Section 5.3 and Section 5.4).
We finally give the q-nonuniforn version of the Lyapunov ntegral inequality by using the q-nonuniform Sturm-Liouville equation in Section 5.5.

In Chapter 6, we present a recent developpement of q-nonuniform difference classical analysis which is the q-nonuniform difference Laplace transform. This is a generalization of the Z-transform in difference analysis [27] and q-Laplace transform in q-difference analysis [17].

Chapter 1

q-Nonuniform difference and integral calculus

In the following lines, we first introduce basic concepts of q-nonuniform difference calculus necessary for the sequel. The functions that are considered in the q-nonuniform difference calculus are clearly defined on the set

$$\mathbb{T} = \Big\{ x(q^k), \quad k \in \frac{\mathbb{Z}}{2} = \{\frac{n}{2}\}_{n\in\mathbb{Z}} \Big\}, \tag{1.1}$$
$$x(z) = \frac{z + z^{-1}}{2}.$$

1.1 q-Nonuniform difference calculus

Here for clarity, we outline basic formula of the differentiation on q-nonuniform lattices. This means that setting $z = q^s$ in (8), the current variable is now $x(z) = \frac{z+z^{-1}}{2}$

1. **Derivative of a product**

$$\begin{aligned}\mathcal{D}(fg)(x(z)) &= f(x(q^{\frac{1}{2}}z))\mathcal{D}g(x(z)) + g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) \\ &= g(x(q^{\frac{1}{2}}z))\mathcal{D}f(x(z)) + f(x(q^{-\frac{1}{2}}z))\mathcal{D}g(x(z)). \qquad (1.2)\end{aligned}$$

2. **Derivative of a ratio**

$$\mathcal{D}(\frac{f}{g}) = \frac{g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) - f(x(q^{-\frac{1}{2}}z))\mathcal{D}g(x(z))}{g(x(q^{\frac{1}{2}}z))g(x(q^{-\frac{1}{2}}z))}. \qquad (1.3)$$

3. **Derivative of a composite function**

$$\begin{gathered}\mathcal{D}(f(g))(x(z)) = \\ \frac{f(g(x(q^{\frac{1}{2}}z))) - f(g(x(q^{-\frac{1}{2}}z)))}{g(x(q^{\frac{1}{2}}z)) - g(x(q^{-\frac{1}{2}}z))}.\frac{g(x(q^{\frac{1}{2}}z)) - g(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \\ = \mathcal{D}_g f.\mathcal{D}_x g.\end{gathered}$$

Hence

$$\mathcal{D}(f(g))(x(z)) \stackrel{\text{def}}{=} \mathcal{D}_g f.\mathcal{D}_x g \qquad (1.4)$$

4. **Derivative of the inverse function**

Let $y = f(x)$. Then $x = f^{-1}(y)$, where f^{-1} is the inverse to f function. Applying the divided difference derivative on both sides of the preceding equation, one obtains

$$\begin{aligned}1 &= \frac{f^{-1}(y(x(q^{\frac{1}{2}}z))) - f^{-1}(y(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \\ &= \frac{f^{-1}(y(x(q^{\frac{1}{2}}z))) - f^{-1}(y(x(q^{-\frac{1}{2}}z))}{y(x(q^{\frac{1}{2}}z)) - y(x(q^{-\frac{1}{2}}z))}.\frac{y(x(q^{\frac{1}{2}}z)) - y(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \\ &= \mathcal{D}_y f^{-1}.\mathcal{D}_x y.\end{aligned}$$

Hence

$$\mathcal{D}_y f^{-1} \stackrel{\text{def}}{=} \frac{1}{\mathcal{D}_x y} \tag{1.5}$$

5. **Derivative of a polynomial**

   Let $P_n(x(z))$ be a polynomial in the variable $x(z) = \dfrac{z+z^{-1}}{2}$. We calculate its derivative to make sure that it is polynomial in $x(z)$ with moreover a degree equal to $n-1$. Using the fact that any polynomial of degree k in $x(z) = \frac{z+z^{-1}}{2}$ can be written as

$$P_k(x(z)) = \sum_{i=0}^{k} a_i \Big(\frac{z+z^{-1}}{2}\Big)^i = \sum_{j=0}^{k} b_j (z^j + z^{-j}) \tag{1.6}$$

   we obtain

$$\begin{aligned}
\mathcal{D}P_n(x(z)) &= \frac{\sum_{j=0}^{n} b_j (q^{\frac{j}{2}} z^j + q^{-\frac{j}{2}} z^{-j}) - \sum_{j=0}^{n} b_j (q^{-\frac{j}{2}} z^j + q^{\frac{j}{2}} z^{-j})}{\frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{2}(z - z^{-1})} \\
&= \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \frac{z^j - z^{-j}}{z - z^{-1}} \\
&= \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \sum_{k=0}^{j-1} c_k (z^k + z^{-k}) \\
&= \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \tilde{P}_{j-1}\Big(\frac{z+z^{-1}}{2}\Big) \\
&= \tilde{\tilde{P}}_{n-1}\Big(\frac{z+z^{-1}}{2}\Big).
\end{aligned}$$

1.2 q-Nonuniform integral calculus

1. **Integration**

We consider the q-nonuniform divided difference derivative defined by (from now on $q \in \mathbb{R}^+, s \in \mathbb{Z}^+$, as indicated below)

$$\mathcal{D}f(x(z)) \stackrel{\text{def}}{=} \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)}, \tag{1.7}$$

$$\begin{aligned} x(z) &= \frac{z + z^{-1}}{2} \\ \Rightarrow x(q^s) &= \frac{q^s + q^{-s}}{2}, \quad q \in \mathbb{R}^+, s \in \mathbb{Z}^+. \end{aligned} \tag{1.8}$$

A function $f(x(z))$ is said to be *q-nonuniform differentiable* on $x(z)$ iff the ratio in the r.h.s. (right hand side) of (1.7) exists and is finite. Clearly, every continuous function on $\mathbb{T}$ (in the topology of $\mathbb{R}$) is q-nonuniform differentiable on that set.

Let's suppose that $\mathcal{D}F(x(z)) = f(x(z)); \quad z = q^s$
or equivalently

$$\frac{F(x(q^{s+\frac{1}{2}})) - F(x(q^{s-\frac{1}{2}}))}{x(q^{s+\frac{1}{2}}) - x(q^{s-\frac{1}{2}})} = f(x(q^s)). \tag{1.9}$$

We have

$$F(x(q^{s-\frac{1}{2}})) - F(x(q^{s+\frac{1}{2}})) = \left[x(q^{s-\frac{1}{2}}) - x(q^{s+\frac{1}{2}})\right] f(x(q^s)),$$

then we have

$$\begin{aligned} F(x(q^s)) - F(x(q^{s+1})) &= \left[x(q^s) - x(q^{s+1})\right] f(x(q^{s+\frac{1}{2}})) \\ F(x(q^{s+1})) - F(x(q^{s+2})) &= \left[x(q^{s+1}) - x(q^{s+2})\right] f(x(q^{s+\frac{3}{2}})) \\ &\vdots \\ F(x(q^{N-1})) - F(x(q^N)) &= \left[x(q^{N-1}) - x(q^N)\right] f(x(q^{N-\frac{1}{2}})). \end{aligned}$$

By adding member by member, we get

$$F(x(q^s)) - F(x(q^N)) = \sum_{r=s}^{N-1} \Big[x(q^r) - x(q^{r+1})\Big] f(x(q^{r+\frac{1}{2}})).$$

Hence

$$\int_{x(q^N)}^{x(z)} f(x(z)) d_q x(z) \stackrel{\text{def}}{=} \sum_{x(t)=x(z)}^{x(q^{N-1})} \big[x(t) - x(qt)\big] f(x(tq^{\frac{1}{2}}))$$
$$= \sum_{t=z}^{q^{N-1}} \big[x(t) - x(qt)\big] f(x(tq^{\frac{1}{2}})), \qquad (1.10)$$

the *q-nonuniform integral* of the function $f(x(z))$ on the interval $[x(q^N), x(z)]$.

This integral sends a polynomial (in $x(z)$) of degree n in a polynomial of degree $n+1$ [13].

Replacing $x(z) = \frac{z+z^{-1}}{2}$ in this last equation we have

$$\int_{x(q^N)}^{x(z)} f(x(z)) d_q x(z) = \frac{1}{2}(1-q) \sum_{t=z}^{q^{N-1}} t(1 - \frac{1}{qt^2}) f(x(tq^{\frac{1}{2}})). \quad (1.11)$$

Let us stop for a moment on the appropriate writing of (1.11) to verify in particular whether in these equations the integral in the left hand side has a lower bound, actually lower than the upper bound. Note first that the function $x(z) = \frac{z+z^{-1}}{2}, \quad z \in \mathbb{R}^+$, is decreasing for $0 < z < 1$ and increasing for $1 < z < \infty$.

Assume first that $0 < q < 1$. In this case, since according to the construction $N \geqslant s \geqslant 0$, one will have $q^N \leqslant z = q^s$ and

$x(z) \leqslant x(q^N)$ and the convenient writting of (1.11) is

$$\int_{x(z)}^{x(q^N)} f(x(z))d_q x(z) = \frac{1}{2}(1-q)\sum_{t=z}^{q^{N-1}} t(\frac{1}{qt^2}-1)f(x(tq^{\frac{1}{2}})). \quad (1.12)$$

On the other hand, if $1 < q < \infty$, we will have $z \leqslant q^N$ and $x(z) \leqslant x(q^N)$ and (1.11) becomes (1.12) again. On other side, in (1.12), the factor

$$h(t) = (1-q)t(\frac{1}{qt^2}-1); \quad t = q^s, \quad s \in \mathbb{Z}^+$$

is always positive regardless of whether $0 < q < 1$ or $1 < q < \infty$ (let us agree from now on that $0 < q < 1$ and therefore $0 \leqslant z \leqslant 1$). This leads us to the following fundamental *positivity* property of the integral in (1.12).

Property 1.1. If $f(x(z)) \geqslant 0$ and $a = x(q^\alpha) \leqslant b = x(q^\beta)$, then

$$\int_a^b f(x(z))d_q x(z) \geqslant 0.$$

Corollary 1.1. *If* $f(x(z)) \geqslant g(x(z))$ *and* $a = x(q^\alpha) \leqslant b = x(q^\beta)$, *then*

$$\int_a^b f(x(z))d_q x(z) \geqslant \int_a^b g(x(z))d_q x(z).$$

2. Connection between the q-integral and q-nonuniform integral

We have

$$\int_{x(z)}^{x(q^N)} f(x(z))d_q x(z) = \frac{1}{2}\int_{q^N}^{z}(\frac{1}{qz^2}-1)f(x(q^{\frac{1}{2}}z))d_q z. \tag{1.13}$$

For $N \to \infty$

$$\begin{aligned}\int_{x(z)}^{\infty} f(x(z))d_q x(z) &= \frac{1}{2}\int_{0}^{z}(\frac{1}{qz^2}-1)f(x(q^{\frac{1}{2}}z))d_q z \\ &= \frac{1}{2}(1-q)z\sum_{i=0}^{\infty}(\frac{1}{q^{2i+1}z^2}-1)f(x(q^{i+\frac{1}{2}}z))\end{aligned} \tag{1.14}$$

and the integral with lower and upper finite bound can be written

$$\int_{a}^{b} f(x(z))d_q x(z) = \int_{a}^{\infty} f(x(z))d_q x(z) - \int_{b}^{\infty} f(x(z))d_q x(z) \tag{1.15}$$

$a = x(q^\alpha) \leqslant b = x(q^\beta)$ (if the integrals at the r.h.s. of (1.15) exist).

A function $f(x(z))$ defined on $\mathbb{T}$ is said to be *q-nonuniform integrable* on a finite interval $[x(z), x(q^N)]$ iff the sum in the r.h.s. of (1.12) exists and is finite. If the upper bound is infinite, the q-nonuniform integrability means the convergence of the infinite series in the r.h.s. of (1.14). Every continuous function on $\mathbb{T}$ (in the topology of $\mathbb{R}$) is clearly q-nonuniform integrable on any finite interval on that set.

Remark 1.2. It is not difficult to notice that to deal with the case where $s \geqslant N \geqslant 0$ it would suffice to replace in the preceding formulas $z = q^s$ by q^N and vice versa.

In this case, from (1.10), we obtain:

$$\int_{x(z)}^{x(q^N)} f(x(z))d_q x(z) \overset{\text{def}}{=} \sum_{x(t)=x(q^N)}^{x(zq^{-1})} \Big[x(t) - x(qt)\Big] f(x(q^{\frac{1}{2}}t))$$

$$= \sum_{t=q^N}^{zq^{-1}} \Big[x(t) - x(qt)\Big] f(x(q^{\frac{1}{2}}t)) \qquad (1.16)$$

or by replacing $x(z) = \frac{z+z^{-1}}{2}$ in (1.16)

$$\int_{x(z)}^{x(q^N)} f(x(z))d_q x(z) = \frac{1}{2}(1-q) \sum_{t=q^N}^{zq^{-1}} t(1 - \frac{1}{qt^2}) f(x(q^{\frac{1}{2}}t)). \quad (1.17)$$

Passing to the q-integral, we will have

$$\int_{x(z)}^{x(q^N)} f(x(z))d_q x(z) = \frac{1}{2} \int_{q^N}^{z} (\frac{1}{qt^2} - 1) f(x(q^{\frac{1}{2}}t))d_q t. \qquad (1.18)$$

For $N \to 0$

$$\int_{1}^{x(z)} f(x(z))d_q x(z) = \frac{1}{2}(q-1) \sum_{t=q^{-1}z}^{1} t(1 - \frac{1}{qt^2}) f(x(q^{\frac{1}{2}}t))$$

$$= \frac{1}{2}(q-1) \sum_{i=0}^{s-1} q^i (1 - \frac{1}{q^{2i+1}}) f(x(q^{i+\frac{1}{2}}))$$

$$= \frac{1}{2} \int_{1}^{z} (1 - \frac{1}{qt^2}) f(x(q^{\frac{1}{2}}t))d_q t. \qquad (1.19)$$

Finally, if $a = x(q^{\alpha}) \leqslant b = x(q^{\beta})$,

$$\begin{aligned}\int_a^b f(x(z))d_q x(z) &\stackrel{\text{def}}{=} \int_1^b f(x(z))d_q x(z) - \int_1^a f(x(z))d_q x(z) \\ &= \int_a^1 f(x(z))d_q x(z) - \int_b^1 f(x(z))d_q x(z).\end{aligned} \tag{1.20}$$

3. **Fundamental principles of analysis**

(**i**) We can formulate the statement of the fundamental principle of analysis as follows: The q-nonuniform derivative of the integral of a function is this function itself. This corresponds to the formula:

$$\begin{aligned}&\mathcal{D}\Big[\int_{x(q^N)}^{x(z)} f(x(z))d_q x(z)\Big] \\ =\ &\mathcal{D}\Big[\sum_{x(q^{\frac{1}{2}}z)}^{x(q^{N-\frac{1}{2}})} [x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z)]f(x(z))\Big] \\ =\ &\frac{\Big[\sum_{x(z)}^{x(q^{N-\frac{1}{2}})} - \sum_{x(qz)}^{x(q^{N-\frac{1}{2}})}\Big]\Big[x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z)\Big]f(x(z))}{x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z)} \\ =\ &f(x(z))\end{aligned}$$

Hence

$$\mathcal{D}\Big[\int_{x(q^N)}^{x(z)} f(x(z))d_q x(z)\Big] = f(x(z)). \tag{1.21}$$

(**ii**) This is the q-nonuniform version of the Newton-Leibnitz formula:

$$\begin{aligned}
\int_{x(q^N)}^{x(z)} \Big[\mathcal{D}f\Big](x(z))d_q x(z) &= \sum_{z(q^{\frac{1}{2}}z)}^{x(q^{N-\frac{1}{2}})} [x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z)](\mathcal{D}f)(x(z)) \\
&= \sum_{z(q^{\frac{1}{2}}z)}^{x(q^{N-\frac{1}{2}})} [f(x(q^{-\frac{1}{2}}z)) - f(x(q^{\frac{1}{2}}z))] \\
&= f(x(z)) - f(x(qz)).
\end{aligned}$$

Hence

$$\int_{x(q^N)}^{x(z)} \Big[\mathcal{D}f\Big](x(z))d_q x(z) = f(x(z)) - f(x(q^N)) \qquad (1.22)$$

4. **Integration by parts**

By integrating the formula

$$f(x(q^{\frac{1}{2}}z))\mathcal{D}g(x(z)) = \mathcal{D}\Big[fg\Big](x(z)) - g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) \tag{1.23}$$

and using the second fundamental principle of the analysis, one obtains

$$\int_{x(q^N)}^{x(z)} \Big[f(x(q^{\frac{1}{2}}z))\mathcal{D}g(x(z))\Big] d_q x(z) =$$

$$\Big[fg\Big]_{x(q^N)}^{x(z)} - \int_{x(q^N)}^{x(z)} g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) d_q x(z). \tag{1.24}$$

5. **Integral of a polynomial**

We now calculate the integral of a polynomial $P_n(x(z))$ of degree n in the variable $x(z) = \frac{z+z^{-1}}{2}$ and make sure that it is a polynomial in $x(z)$ with moreover a degree equal to $n+1$. The relation (1.6) will also be used. So, for a given polynomial $P_n(x(z))$, we search a function $f(x(z))$ such that

$$\mathcal{D}f(x(z)) = \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} = P_n(x(z)) \tag{1.25}$$

Hence

$$\begin{aligned} f(x(q^{-\frac{1}{2}}z)) - f(x(q^{\frac{1}{2}}z)) &= x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z)P_n(x(z)) \\ &= \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{2}(z - z^{-1})P_n\Big(\frac{z + z^{-1}}{2}\Big) \\ &= \sum_{j=1}^{n+1} a_j(z^j - z^{-j}) \\ &= P^1_{n+1}(z) + P^2_{n+2}(z) \end{aligned}$$

where

$$P^1_{n+1}(z) = \sum_{i=1}^{n+1} a_i z^i, \qquad P^2_{n+1}(z) = -\sum_{i=1}^{n+1} a_i z^{-i} \tag{1.26}$$

Now, let us consider two functions $f_1(x(z))$ and $f_2(x(z))$ such that $f(x(z)) = f_1(x(z)) + f_2(x(z))$ and

$$f_1(x(q^{-\frac{1}{2}}z)) - f_1(x(q^{\frac{1}{2}}z)) = P^1_{n+1}(z) \tag{1.27}$$

$$f_2(x(q^{-\frac{1}{2}}z)) - f_2(x(q^{\frac{1}{2}}z)) = P^2_{n+1}(z) \tag{1.28}$$

From (1.27) and (1.28) it follows, respectively, that

$$f_1(x(z)) = \sum_{i=0}^{\infty} P^1_{n+1}(q^{i+\frac{1}{2}}z) + c_1 = \sum_{i=1}^{n+1} \frac{a_i q^{\frac{i}{2}}}{1 - q^i} z^i + c_1 \tag{1.29}$$

$$f_2(x(z)) = -\sum_{i=0}^{\infty} P^2_{n+1}(q^{-i-\frac{1}{2}}z) + c_2 = \sum_{i=1}^{n+1} \frac{a_i q^{\frac{i}{2}}}{1 - q^i} z^{-i} + c_2 \tag{1.30}$$

Hence

$$\begin{aligned} f(x(z)) &= f_1(x(z)) + f_2(x(z)) \\ &= c + \sum_{i=1}^{n+1} \frac{a_i q^{\frac{i}{2}}}{1-q^i}(z^i + z^{-i}) \\ &= \tilde{P}\Big(\frac{z+z^{-1}}{2}\Big). \end{aligned} \tag{1.31}$$

Chapter 2

q-Nonuniform difference equations

2.1 q-Nonuniform difference equations of first order and elementary functions

2.1.1 Linear q-nonuniform difference equations of first order

Consider the first order q-nonuniform difference equation

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)) + b(x(z)). \tag{2.1}$$

As in differential or difference calculus, to solve it, we first solve the corresponding homogeneous equation

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)). \tag{2.2}$$

Developping it

$$\frac{y_0(x(zq^{\frac{1}{2}}) - y_0(x(zq^{-\frac{1}{2}})}{x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}})} = a(x(z))y_0(x(zq^{-\frac{1}{2}})),$$

we get

$$y_0(x(zq^{\frac{1}{2}})) = p(x(z))y_0(x(zq^{-\frac{1}{2}})) \tag{2.3}$$

where

$$\begin{aligned} p(x(z)) &= 1 + (x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))a(x(z)) \\ &= 1 + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}(z - z^{-1})a(x(z)). \end{aligned} \tag{2.4}$$

By the recursion

$$y_0(x(z)) = [p(x(zq^{\frac{1}{2}}))]^{-1}y_0(x(zq)), \tag{2.5}$$

we get

$$y_0(x(z)) = y_0(x(zq^N)) \prod_{i=0}^{N-1} \left[p(x(zq^{\frac{1}{2}+i}))\right]^{-1}. \tag{2.6}$$

Consider now the following equation

$$\mathcal{D}\tilde{y}(x(z)) = \tilde{a}(x(z))\tilde{y}(x(zq^{\frac{1}{2}})) + \tilde{b}(x(z)) \tag{2.7}$$

and its homogenous part

$$\mathcal{D}\tilde{y}(x(z)) = \tilde{a}(x(z))\tilde{y}(x(zq^{\frac{1}{2}})). \tag{2.8}$$

Developping the latter, we get the relation

$$\tilde{y}(x(zq^{\frac{1}{2}}))\tilde{p}(x(z)) = \tilde{y}(x(zq^{-\frac{1}{2}})). \tag{2.9}$$

where

$$\begin{aligned}\tilde{p}(x(z)) &= 1 - (x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))a(x(z)) \\ &= 1 - \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}(z - z^{-1})\tilde{a}(x(z)). \end{aligned} \tag{2.10}$$

Using the recursion

$$\tilde{y}_0(x(z)) = \tilde{p}(x(zq^{\frac{1}{2}}))\tilde{y}_0(x(zq)), \tag{2.11}$$

we get

$$\tilde{y}_0(x(z)) = \tilde{y}_0(x(z_0)) \prod_{t=z}^{q^{-1}z_0} \tilde{p}(x(tq^{\frac{1}{2}})), \tag{2.12}$$

or

$$\tilde{y}_0(x(z)) = \left(\prod_{i=0}^{N-1} \tilde{p}(x(zq^{\frac{1}{2}+i})) \right) \tilde{y}_0(x(zq^N)). \tag{2.13}$$

Clearly, if $\tilde{a} = -a$, then

$$\tilde{y}_0(x(z)).y_0(x(z)) = \tilde{y}_0(x(q^N z)).y_0(x(q^N z)). \tag{2.14}$$

Consequently, if for certain N, $\tilde{y}_0(x(q^N z)).y_0(x(q^N z)) = 1$, then the two solutions are mutually inverse.

More generally, we have the following

Theorem 2.1. *If $y(x(z))$ and $\tilde{y}(x(z))$ are respective solutions of the equations*

$$\begin{aligned}\mathcal{D}y(x(z)) &= a(x(z))y(x(zq^{-\frac{1}{2}})), \\ \mathcal{D}\tilde{y}(x(z)) &= -a(x(z))\tilde{y}(x(zq^{-\frac{1}{2}})), \end{aligned} \tag{2.15}$$

satifying the conditions

$$y(x(z_0))\tilde{y}(x(z_0)) = 1, \tag{2.16}$$

then

$$y(x(z))\tilde{y}(x(z)) = 1. \tag{2.17}$$

Proof.

$$\begin{aligned} \mathcal{D}(y\tilde{y}) &= y(x(q^{-\frac{1}{2}}z))\mathcal{D}\tilde{y} + \tilde{y}(x(zq^{\frac{1}{2}}))\mathcal{D}y \\ &= y(x(q^{-\frac{1}{2}}z))(-a)\tilde{y}(x(zq^{\frac{1}{2}})) + \tilde{y}(x(zq^{\frac{1}{2}}))(a)y(x(q^{-\frac{1}{2}}z)) \\ &= 0. \end{aligned}$$

This means that $y\tilde{y} = const$, which, by (2.16) gives (2.17). □

To ensure the exponential shape of the solution of (2.2), we solve it *heuristically* as follows. First, we write it in the form of a q-difference equation in the z-variable:

$$D_q y(x(z)) = \gamma(z)y(x(z)), \tag{2.18}$$

where

$$\gamma(z) = \frac{1}{2}a(x(q^{\frac{1}{2}}z))(1 - \frac{1}{qz^2}). \tag{2.19}$$

To deal with (2.18), define a q-version of the logarithm function (of function as variable) by setting

$$(\widetilde{\ln}_q f)(z) =^{def} \int_{z_0}^{z} \frac{D_q f(z)}{f(z)} d_q z, \tag{2.20}$$

where $z_0 = f^{-1}(1)$; and a q-version of the exponential function as the inverse of the q-logarithm

$$\tilde{e}_q =^{def} \widetilde{\ln}_q^{-1}. \tag{2.21}$$

This means that

$$f(z) = \tilde{e}_q^{\int_{z_0}^{z} \frac{D_q f(z)}{f(z)} d_q z}. \tag{2.22}$$

Examples.

1. By (2.20)

$$\begin{aligned} \widetilde{\ln}_q z^b &= \int_1^z \frac{D_q z^b}{z^b} d_q z = \frac{q^b - 1}{q - 1} \int_1^z \frac{1}{z} d_q z \\ &= \frac{q^b - 1}{\ln q} \ln_q z = \log_q z^{q^b - 1} \end{aligned} \tag{2.23}$$

2. If $e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}$ is the usual q-exponential function, then by (2.22) we have that

$$D_q e_q^{az} = a e_q^{az} \Rightarrow e_q^{az} = \tilde{e}_q^{\int_0^z a d_q z} = \tilde{e}_q^{az}, \tag{2.24}$$

and

$$D_q e_{q^2}^{z^2} = (q+1) z e_{q^2}^{z^2} \Rightarrow e_{q^2}^{z^2} = \tilde{e}_q^{\int_0^z (q+1) z d_q z} = \tilde{e}_q^{z^2}. \tag{2.25}$$

More generally

$$D_q e_{q^b}^{z^b} = (D_q z^b) e_{q^b}^{z^b} \Rightarrow e_{q^b}^{z^b} = \tilde{e}_q^{\int_0^z (D_q z^b) d_q z} = \tilde{e}_q^{z^b}. \tag{2.26}$$

On the other side by (2.24), (2.25) and (2.26), we have that

$$\widetilde{\ln}_q(e_q^{az}) = az = \ln_q(e_q^{az}), \tag{2.27}$$

$$\widetilde{\ln}_q(e_{q^2}^{z^2}) = z^2 \neq \ln_q(e_{q^2}^{z^2}), \tag{2.28}$$

$$\widetilde{\ln}_q(e_{q^b}^{z^b}) = z^b \neq \ln_q(e_{q^b}^{z^b}), \tag{2.29}$$

where $\ln_q$ is the inverse to usual q-exponential e_q.

It is also worth noting that unlike the usual q-exponential and q-logarithm cases, from (2.20) and (2.22), follow clearly the formulae

$$D_q\widetilde{\ln}_q u(z) = \frac{D_q u(z)}{u(z)}, \;\; D_q\tilde{e}_q^{v(z)} = \tilde{e}_q^{v(z)} D_q v(z) \tag{2.30}$$

similarly to the differential situations.

We can now write *formally* the solution of (2.18)-(2.19) as

$$y(x(z)) = \tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = \tilde{e}_q^{\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(D_q x(z))d_q z}. \tag{2.31}$$

To compare (2.6) with (2.31), let us write the former as

$$y_0(x(z)) = [\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}})]^{-1} y_0(x(z_0)), \;\; z = z_0.q^{-k_z} \tag{2.32}$$

for some positive integer k_z and complex $z_0 \neq 0$. Hence one can write

$$\tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = \left(\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}})\right)^{-1}, \quad z = z_0.q^{-k_z}, \tag{2.33}$$

where we deleted the factor $y_0(x(z_0))$, since in the definition (2.20), z_0 was chosen so that $y(x(z_0)) = 1$.

If $z_0 = 0$ then

$$\tilde{e}_q^{\frac{1}{2}\int_0^z a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = [\prod_{i=0}^{\infty} p(q^{\frac{1}{2}+i}z)]^{-1} \tag{2.34}$$

provided the convergence of the involved integral and product. It is not difficult to see that the convergence of the product in right is linked to that of the corresponding q-integral in lhs. Indeed, if $h(z) = \frac{1}{2}a(x(zq^{\frac{1}{2}}))(1 - \frac{1}{qz^2})$ is the integrated function in lhs of (2.34), then $p(q^{\frac{1}{2}}z) = (q-1)zh(z) + 1$ and consequently, the convergence of the product in (2.34) is equivalent to that of the series $\sum_{i=0}^{\infty}(q-1)zq^i h(zq^i) = -\int_0^z h(z)d_q z$.

Consider now the non homogeneous equation (2.1). $y_0(x(z))$ being the solution of the corresponding homogeneous equation (2.2), we are led to search its solution by the *method of variation of constants* under the form

$$y(x(z)) = c(x(z))y_0(x(z)), \tag{2.35}$$

for unknown $c(x(z))$. Loading (2.35) in (2.1) gives

$$c(x(z)) = c + \int_{x(z_0)}^{x(z)} y_0^{-1}(x(q^{\frac{1}{2}}t))b(x(t))d_q x(t). \tag{2.36}$$

By (2.35) we then get the general form of the solution of (2.1):

$$y(x(z)) = \phi(x(z), x(z_0))[y(x(z_0)) + \int_{x(z_0)}^{x(z)} \phi(x(z_0), x(q^{\frac{1}{2}}t))b(x(t))d_q x(t)], \tag{2.37}$$

where

$$\phi(a,b) = y_0(a)y_0^{-1}(b). \tag{2.38}$$

For the associated equation (2.7), the general solution reads

$$\tilde{y}(x(z)) = \tilde{\phi}(x(z),x(z_0))[\tilde{y}(x(z_0)) + \int_{x(z_0)}^{x(z)} \tilde{\phi}(x(z_0),x(q^{-\frac{1}{2}}t))\tilde{b}(x(t))d_qx(t)], \tag{2.39}$$

where now

$$\tilde{\phi}(a,b) = \tilde{y}_0(a)\tilde{y}_0^{-1}(b). \tag{2.40}$$

We will now consider some special cases of equation (2.1). Since by (2.37), the essential of the solution of (2.1) consists in solving (2.2), we can consider here only homogeneous cases:

Case 1.

$$\mathcal{D}y(x(z)) = ay(x(q^{-\frac{1}{2}}z)), \tag{2.41}$$

a-const. We can solve this equation in two ways: First *formally*, we simply use (2.31) to get

$$y(x(z)) = \tilde{e}_q^{\frac{a}{2}\int_{z_0}^{z}(1-\frac{1}{qz^2})d_qz} = \tilde{e}_q^{a\int_{z_0}^{z}(D_qx(z))d_qz} = \tilde{e}_q^{\frac{a}{2}(z+z^{-1})} = \tilde{e}_q^{ax(z)} \tag{2.42}$$

as a solution of (2.41), $y(z_0) = 1$ for $z_0 = \pm i$.

To solve (2.41) *explicitly*, we use (2.5). Factorizing $p(q^{1/2}z)$ with $a(x(z)) = a$, (2.5) gives

$$y_0(x(z)) = c\frac{1}{(1 - z_1 z^{-1})(1 - z z_2^{-1})} y_0(x(qz)) \tag{2.43}$$

where

$$c = \frac{2}{(1-q)az_2};\ z_{1,2} = \frac{-q \pm [q^2 + q(q-1)^2 a^2]^{\frac{1}{2}}}{aq(q-1)} \tag{2.44}$$

and whose solution reads

$$y_0(x(z)) = z^{\frac{\ln c}{\ln q}} \frac{(z_1 q z^{-1}; q)_\infty}{(z_2^{-1} z; q)_\infty}, \tag{2.45}$$

where $(\alpha; q)_n = \prod_{i=0}^{n-1} (1 - q^i \alpha)$ and $(\alpha; q)_\infty = \lim_{n\to\infty} (\alpha; q)_n$.

Comparing (2.42) and (2.45), one obtains an explicit expression for $\tilde{e}_q^{ax(z)}$:

$$\tilde{e}_q^{ax(z)} = K(z_0) z^{\frac{\ln c}{\ln q}} \frac{(z_1 q z^{-1}; q)_\infty}{(z_2^{-1} z; q)_\infty} \tag{2.46}$$

where $K(z_0)$ is a constant determined by the condition $\tilde{e}_q^{ax(z_0)} = 1$

Case 2.

$$\mathcal{D} y(x(z)) = a.x(z) y(x(q^{-\frac{1}{2}} z)). \tag{2.47}$$

As in the previous case, to solve it in the formal way, we first write it in the form of a q-difference equation in the z-variable:

$$D_q y(x(z)) = \beta(z) y(x(z)), \tag{2.48}$$

where

$$\beta(z) = \frac{a}{4}(q^{\frac{1}{2}}z - q^{\frac{-3}{2}}z^{-3}) = a\frac{q^{\frac{1}{2}}}{q+1}D_q x^2(z), \tag{2.49}$$

and we find

$$y(x(z) = \tilde{e}_q^{\int^z \beta(z)d_q z} = \tilde{e}_q^{\frac{a}{4}\frac{q^{\frac{1}{2}}}{q+1}(z+z^{-1})^2} = \tilde{e}_q^{a\frac{q^{\frac{1}{2}}}{q+1}x^2(z)}, \tag{2.50}$$

again for $z_0 = \pm i$, as the solution of (2.41).

To solve (2.41) explicitly, we also use (2.5). Factorizing $p(q^{1/2}z)$ with $a(x(z)) = a.x(z)$, (2.5) gives

$$y_0(x(z)) = c\frac{1}{(1 - z_1 z^{-2})(1 - z^2 z_2^{-1})}y_0(x(qz)) \tag{2.51}$$

where

$$c = \frac{4}{(1-q)q^{\frac{1}{2}}az_2};\ z_{1,2} = \frac{-2q^{\frac{1}{2}} \pm [4q + (q-1)^2a^2]^{\frac{1}{2}}}{aq(q-1)} \tag{2.52}$$

and whose solution reads

$$y_0(x(z)) = z^{\frac{\ln c}{\ln q}}\frac{(z_1 q^2 z^{-2}; q^2)_\infty}{(z_2^{-1}z^2; q^2)_\infty}. \tag{2.53}$$

Hence

$$\tilde{e}_q^{a\frac{q^{\frac{1}{2}}}{q+1}x^2(z)} = K(z_0)z^{\frac{\ln c}{\ln q}}\frac{(z_1 q^2 z^{-2}; q^2)_\infty}{(z_2^{-1}z^2; q^2)_\infty} \tag{2.54}$$

where $K(z_0)$ is determined by the condition $\tilde{e}_q^{a\frac{q^{\frac{1}{2}}}{q+1}x^2(z_0)} = 1$.

Case 3.

$$\mathcal{D}y(x(z)) = \frac{1}{x(zq^{-\frac{1}{2}}) + \nu} y(x(q^{-\frac{1}{2}}z)). \tag{2.55}$$

Using (2.31), we get

$$y(x(z)) = \tilde{e}_q^{\int_{z_0}^{z} \frac{D_q(x(z)+\nu)}{x(z)+\nu} d_q z} = \tilde{e}_q^{\widetilde{\ln}_q(x(z)+\nu)} = x(z) + \nu, \tag{2.56}$$

while $x(z_0) + \nu = 1$.

Case 4.

$$\mathcal{D}y(x(z)) = \frac{q+1}{q^{\frac{1}{2}}} \frac{x(z)}{x^2(zq^{-\frac{1}{2}}) + \mu} y(x(q^{-\frac{1}{2}}z)). \tag{2.57}$$

Again, using (2.31), we get

$$y(x(z)) = \tilde{e}_q^{\int_{z_0}^{z} \frac{D_q(x^2(z)+\mu)}{x^2(z)+\mu} d_q z} = \tilde{e}_q^{\widetilde{\ln}_q(x^2(z)+\mu)} = x^2(z) + \mu, \tag{2.58}$$

while $x^2(z_0) + \nu = 1$.

Case 5.

$$\mathcal{D}y(x(z)) = \frac{a}{x(bz)} y(x(q^{-\frac{1}{2}}z)). \tag{2.59}$$

Here we search the series solution $y(x(z)) = \sum_{n=-\infty}^{+\infty} c_n z^n$. Loading the latter in (2.59) and comparing the coefficients, one gets the easily solvable first order difference equation

$$\tilde{c}_n = \frac{ba(q^{\frac{1}{2}} - q^{-\frac{1}{2}})q^{-n+1} - 2b^2(q^{n-1} - q^{-n+1})}{2(q^n - q^{-n}) + ba(q^{\frac{1}{2}} - q^{-\frac{1}{2}})q^{-n}} \tilde{c}_{n-1};\ \tilde{c}_0, \tag{2.60}$$

where $\tilde{c}_n = c_{2n}$.

Case 6. Similarly to the previous case, one can solve the following equation

$$\mathcal{D}y(x(z)) = \frac{a.x(z)}{x^2(bz) - 2} y(x(q^{-\frac{1}{2}}z)). \tag{2.61}$$

It is useful to note that from the preceding, one deduces directly that the function

$$y(x(z)) = \tilde{e}_{q^{-1}}^{-ax(z)} = \frac{1}{K(z_0)} z^{-\frac{\ln c}{\ln q}} \frac{(z_2^{-1}z;q)_\infty}{(z_1 q z^{-1};q)_\infty}, \tag{2.62}$$

which is the inverse of $\tilde{e}_q^{ax(z)} = z^{\frac{\ln c}{\ln q}} \frac{(z_1 q z^{-1};q)_\infty}{(z_2^{-1}z;q)_\infty}$, is the solution of the equation

$$\mathcal{D}y(x(z)) = -ay(x(q^{\frac{1}{2}}z)). \tag{2.63}$$

Similarly the function $y(x(z)) = \tilde{e}_{q^{-1}}^{-a\frac{q^{\frac{1}{2}}}{q+1}x^2(z)} = \frac{1}{K(z_0)} z^{-\frac{\ln c}{\ln q}} \frac{(z_2^{-1}z^2;q^2)_\infty}{(z_1 q z^{-2};q^2)_\infty}$ is the solution of

$$\mathcal{D}y(x(z)) = -ax(z)y(x(q^{\frac{1}{2}}z)), \tag{2.64}$$

while $y(x(z)) = \frac{1}{x(z)+\nu}$ and $y(x(z)) = \frac{1}{x^2(z)+\mu}$ solve the equations

$$\mathcal{D}y(x(z)) = -\frac{1}{x(zq^{-\frac{1}{2}}) + \nu} y(x(q^{\frac{1}{2}}z)), \tag{2.65}$$

and

$$\mathcal{D}y(x(z)) = -\frac{q+1}{q^{\frac{1}{2}}} \frac{x(z)}{x^2(zq^{-\frac{1}{2}}) + \mu} y(x(q^{\frac{1}{2}}z)) \tag{2.66}$$

respectively.

2.1.2 Nonlinear q-nonuniform difference equations transformable in linear equations

Here, we consider some nonlinear q-nonuniform difference equations transformable in linear equations.

Case 1. Riccati type equations:

$$\mathcal{D}y(x(z)) = a(x)y(x(zq^{\frac{1}{2}})) + b(x)y(x(zq^{-\frac{1}{2}}))y(x(zq^{\frac{1}{2}})). \quad (2.67)$$

To solve this equation, we set $y(x) = 1/u(x)$ and obtain

$$\mathcal{D}u(x(z)) = -[a(x)u(x(zq^{-\frac{1}{2}})) + b(x)]. \quad (2.68)$$

Case 2. Homogenous equations of the form

$$f(\frac{\mathcal{D}y(x)}{y(x)}, x) = 0. \quad (2.69)$$

They can be transformed into linear equations in $u(x)$ with $u(x) = \frac{\mathcal{D}y(x)}{y(x)}$.

Example. Solve the equation

$$[\mathcal{D}y(x)]^2 - 5y(x)\mathcal{D}y(x) + 6[y(x)]^2 = 0. \quad (2.70)$$

To solve the equation, we divide it by $[y(x)]^2$. We get

$$[\frac{\mathcal{D}y(x)}{y(x)}]^2 - 2[\frac{\mathcal{D}y(x)}{y(x)}] - 3 = 0, \quad (2.71)$$

or $u^2(x) - 5u(x) + 6 = 0$, $u(x) = \frac{\mathcal{D}y(x)}{y(x)}$. This gives $u(x) = 3$ and $u(x) = 2$, or $y(x) = ce_q^{3x}$ and $y(x) = ce_q^{2x}$, respectively.

2.1.3 q-Nonuniform exponential and trigonometric functions

Define the exponential function

$$E_{q^{-\frac{1}{2}}}(a;z_0;z) \overset{\text{def}}{=} \begin{cases} \prod_{t=z}^{q^{-1}z_0} \left[p(x(tq^{\frac{1}{2}}))\right]^{-1}, & z > z_0, \\ 1, & z = z_0 > 0. \end{cases} \tag{2.72}$$

Clearly, we have the relation

$$E_{q^{-\frac{1}{2}}}(a;z_0;z) = \left[p(x(zq^{\frac{1}{2}}))\right]^{-1} E_{q^{-\frac{1}{2}}}(a;z_0;qz). \tag{2.73}$$

This means that $E_{q^{-\frac{1}{2}}}(a;z_0;z)$ is a solution of (2.11) and consequently, that of (2.5). Morever,

$$E_{q^{-\frac{1}{2}}}(a;z_0;z) = \left[p(x(z_0q^{-\frac{1}{2}}))\right]^{-1} E_{q^{-\frac{1}{2}}}(a;q^{-1}z_0;z). \tag{2.74}$$

From the definition (2.72), we can evaluate

$$E_{q^{-\frac{1}{2}}}(a;0;z) = \prod_{i=0}^{\infty} \left[p(x(q^i z q^{\frac{1}{2}}))\right]^{-1}, \tag{2.75}$$

and

$$E_{q^{-\frac{1}{2}}}(a;z_0;\infty) = \prod_{i=0}^{\infty} \left[p(x(q^{-i}q^{-1}z_0q^{\frac{1}{2}}))\right]^{-1} = \prod_{i=-1}^{-\infty} \left[p(x(q^i z_0 q^{\frac{1}{2}}))\right]^{-1} \tag{2.76}$$

Also, one easily verifies that $E_{q^{-\frac{1}{2}}}(a;z_0;\infty)$ is a solution of (2.74) for $z = \infty$.

If we set $z = z_0 = 1$ in (2.75) and (2.76), we get

$$E_{q^{-\frac{1}{2}}}(a;0;\infty) = \prod_{i=-\infty}^{\infty} \left[p(x(q^i q^{\frac{1}{2}}))\right]^{-1}. \tag{2.77}$$

On the other side, since $0 < q < 1$, then, for $a(x(z)) = a = const$, $z_0, z \neq 0, \infty$, one can verify that

$$E_{q^{-\frac{1}{2}}}(a; 0; z) = 0 = E_{q^{-\frac{1}{2}}}(a; z_0; \infty). \tag{2.78}$$

Indeed,

$$\begin{aligned} p(x(zq^{\frac{1}{2}})) &= 1 + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}(q^{\frac{1}{2}}z - q^{-\frac{1}{2}}z^{-1})a \\ &= \frac{a(q-1)}{2}\frac{(z-z_1)(z-z_2)}{z}; \end{aligned} \tag{2.79}$$

with

$$\begin{aligned} z_1 &= \frac{1 - 2q + 2\sqrt{q^2 - 2a^2q^2 + a^2q^3 + a^2q}}{2(aq^2 - aq)}, \\ z_2 &= \frac{1 - 2q - 2\sqrt{q^2 - 2a^2q^2 + a^2q^3 + a^2q}}{2(aq^2 - aq)}. \end{aligned}$$

Hence

$$\begin{aligned} E_{q^{-\frac{1}{2}}}(a; 0; z) &= \prod_{i=0}^{\infty}\left[p(x(q^i z q^{\frac{1}{2}}))\right]^{-1} \\ = \frac{2z}{a(q-1)z_1z_2}\frac{1}{(z_1^{-1}z; q)_\infty (z_2^{-1}z; q)_\infty}.\prod_{i=0}^{\infty} q^i &= 0, \end{aligned} \tag{2.80}$$

where $(u; q)_\infty \stackrel{\text{def}}{=} \prod_{i=0}^{\infty}(1-uq^i)$. Similarly, one proves that $E_{q^{-\frac{1}{2}}}(a; z_0; \infty) = 0$.

Define the other exponential function

$$E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z) \stackrel{\text{def}}{=} \begin{cases} \prod_{t=z}^{q^{-1}z_0} \tilde{p}(x(tq^{\frac{1}{2}})), & z > z_0, \\ 1, & z = z_0 > 0. \end{cases} \tag{2.81}$$

Here also, the function $E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z)$ admet representations and properties similar to that of $E_{q^{-\frac{1}{2}}}(a; z_0; z)$. In particular, $E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z)$

verifies the relation

$$E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z) = \tilde{p}(x(zq^{\frac{1}{2}}))E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; zq). \tag{2.82}$$

This means that $E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z)$ is a solution of (2.11) and consequently, that of (2.8). Morever,

$$E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z) = \tilde{p}(x(z_0q^{-\frac{1}{2}}))E_{q^{\frac{1}{2}}}(\tilde{a}; q^{-1}z_0; z). \tag{2.83}$$

Similarly, we evaluate

$$E_{q^{\frac{1}{2}}}(\tilde{a}; 0; z) = \prod_{i=0}^{\infty} p(x(q^i zq^{\frac{1}{2}})), \tag{2.84}$$

and

$$E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; \infty) = \prod_{i=-1}^{-\infty} \tilde{p}(x(q^i z_0 q^{\frac{1}{2}})). \tag{2.85}$$

Also, one easily verifies that $E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; \infty)$ is a solution of (2.83) for $z = \infty$.

Also

$$E_{q^{\frac{1}{2}}}(\tilde{a}; 0; \infty) = \prod_{i=-\infty}^{\infty} \tilde{p}(x(q^i z_0 q^{\frac{1}{2}})), \tag{2.86}$$

and

$$E_{q^{\frac{1}{2}}}(\tilde{a}; 0; z) = \infty = E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; \infty). \tag{2.87}$$

Consider now the product $h(z) = E_{q^{-\frac{1}{2}}}(a; z_0; z).E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; z)$. Define $K_{a,\tilde{a}}$ by

$$\begin{aligned} K_{a,\tilde{a}} = h(\infty) &= E_{q^{-\frac{1}{2}}}(a; z_0; \infty).E_{q^{\frac{1}{2}}}(\tilde{a}; z_0; \infty) \\ &= \prod_{i=1}^{\infty} \frac{\tilde{p}(x(q^{-i}z_0q^{\frac{1}{2}}))}{p(x(q^{-i}z_0q^{\frac{1}{2}}))}. \end{aligned} \tag{2.88}$$

Simple computations give

$$K_{a,\tilde{a}} = \frac{-\tilde{a}}{a}\frac{(1-z_0^{-1}z_1)(1-z_0^{-1}z_2)}{(1-z_0^{-1}\tilde{z}_1)(1-z_0^{-1}\tilde{z}_2)}\frac{(z_0^{-1}\tilde{z}_1;q)_\infty(z_0^{-1}\tilde{z}_2;q)_\infty}{(z_0^{-1}z_1;q)_\infty(z_0^{-1}z_2;q)_\infty}, \tag{2.89}$$

where z_1, z_2 and $\tilde{z}_1, \tilde{z}_2$ are finite roots of $p(q^{\frac{1}{2}}z) = 0$ and $\tilde{p}(q^{\frac{1}{2}}z) = 0$, respectively, knowing that $\tilde{p}(x(z))$ is obtained from $p(x(z))$ by replacing a by $-\tilde{a}$.

From (2.4), (2.72), (2.10) and (2.81), we get the following special case of theorem 2.1.

Theorem 2.2.

$$E_{q^{-\frac{1}{2}}}(a; z_0; z).E_{q^{\frac{1}{2}}}(-a; z_0; z) = 1 = E_{q^{-\frac{1}{2}}}(-a; z_0; z).E_{q^{\frac{1}{2}}}(a; z_0; z) \tag{2.90}$$

Consider next the following definitions

1.

$$\begin{aligned} \cos_{q^{\frac{1}{2}}}(a; z_0; z) &\stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(ai; z_0; z) + E_{q^{\frac{1}{2}}}(-ai; z_0; z)}{2}; \\ \sin_{q^{\frac{1}{2}}}(a; z_0; z) &\stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(ai; z_0; z) - E_{q^{\frac{1}{2}}}(-ai; z_0; z)}{2i}; \\ \cosh_{q^{\frac{1}{2}}}(a; z_0; z) &\stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(a; z_0; z) + E_{q^{\frac{1}{2}}}(-a; z_0; z)}{2}; \\ \sinh_{q^{\frac{1}{2}}}(a; z_0; z) &\stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(a; z_0; z) - E_{q^{\frac{1}{2}}}(-a; z_0; z)}{2}. \end{aligned} \tag{2.91}$$

2.

$$\cos_{q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{-\frac{1}{2}}}(ai;z_0;z)+E_{q^{-\frac{1}{2}}}(-ai;z_0;z)}{2};$$
$$\sin_{q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{-\frac{1}{2}}}(ai;z_0;z)-E_{q^{-\frac{1}{2}}}(-ai;z_0;z)}{2i};$$
$$\cosh_{q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{-\frac{1}{2}}}(a;z_0;z)+E_{q^{-\frac{1}{2}}}(-a;z_0;z)}{2};$$
$$\sinh_{q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{-\frac{1}{2}}}(a;z_0;z)-E_{q^{-\frac{1}{2}}}(-a;z_0;z)}{2}. \qquad (2.92)$$

3.

$$\cos_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(ai;z_0;z)+E_{q^{-\frac{1}{2}}}(-ai;z_0;z)}{2};$$
$$\sin_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(ai;z_0;z)-E_{q^{-\frac{1}{2}}}(-ai;z_0;z)}{2i};$$
$$\cosh_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(a;z_0;z)+E_{q^{-\frac{1}{2}}}(-a;z_0;z)}{2};$$
$$\sinh_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a;z_0;z) \stackrel{\text{def}}{=} \frac{E_{q^{\frac{1}{2}}}(a;z_0;z)-E_{q^{-\frac{1}{2}}}(-a;z_0;z)}{2}. \qquad (2.93)$$

We get

$$\mathcal{D}\cos_{q^{\frac{1}{2}}}(a;z_0;z) = -a\sin_{q^{\frac{1}{2}}}(a;z_0;zq^{\frac{1}{2}});$$
$$\mathcal{D}^-\cos_{q^{\frac{1}{2}}}(a;z_0;z) = -a\sin_{q^{\frac{1}{2}}}(a;z_0;z),$$
$$\mathcal{D}\sin_{q^{\frac{1}{2}}}(a;z_0;z) = a\cos_{q^{\frac{1}{2}}}(a;z_0;zq^{\frac{1}{2}});$$
$$\mathcal{D}^-\sin_{q^{\frac{1}{2}}}(a;z_0;z) = a\cos_{q^{\frac{1}{2}}}(a;z_0;z), \qquad (2.94)$$

where $\mathcal{D}^-f(x(z)) = Df(x(z))\big|_{z:=zq^{-\frac{1}{2}}}$. Hence

$$[\mathcal{D}^-]^2\cos_{q^{\frac{1}{2}}}(a;z_0;z) = -a^2\cos_{q^{\frac{1}{2}}}(a;z_0;z),$$
$$[\mathcal{D}^-]^2\sin_{q^{\frac{1}{2}}}(a;z_0;z) = -a^2\sin_{q^{\frac{1}{2}}}(a;z_0;z). \qquad (2.95)$$

Similarly

$$\begin{aligned}
\mathcal{D}\cosh_{q^{\frac{1}{2}}}(a;z_0;z) &= a\sinh_{q^{\frac{1}{2}}}(a;z_0;zq^{\frac{1}{2}});\\
\mathcal{D}^-\cosh_{q^{\frac{1}{2}}}(a;z_0;z) &= a\sinh_{q^{\frac{1}{2}}}(a;z_0;z),\\
\mathcal{D}\sinh_{q^{\frac{1}{2}}}(a;z_0;z) &= a\cosh_{q^{\frac{1}{2}}}(a;z_0;zq^{\frac{1}{2}});\\
\mathcal{D}^-\sinh_{q^{\frac{1}{2}}}(a;z_0;z) &= a\cosh_{q^{\frac{1}{2}}}(a;z_0;z),
\end{aligned} \tag{2.96}$$

and consequently

$$\begin{aligned}
[\mathcal{D}^-]^2\cosh_{q^{\frac{1}{2}}}(a;z_0;z) &= a^2\cosh_{q^{\frac{1}{2}}}(a;z_0;z),\\
[\mathcal{D}^-]^2\sinh_{q^{\frac{1}{2}}}(a;z_0;z) &= a^2\sinh_{q^{\frac{1}{2}}}(a;z_0;z).
\end{aligned} \tag{2.97}$$

On the other side, we have

$$\begin{aligned}
\mathcal{D}\cos_{q^{-\frac{1}{2}}}(a;z_0;z) &= -a\sin_{q^{-\frac{1}{2}}}(a;z_0;zq^{-\frac{1}{2}});\\
\mathcal{D}^+\cos_{q^{-\frac{1}{2}}}(a;z_0;z) &= -a\sin_{q^{-\frac{1}{2}}}(a;z_0;z),\\
\mathcal{D}\sin_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\cos_{q^{-\frac{1}{2}}}(a;z_0;zq^{-\frac{1}{2}});\\
\mathcal{D}^+\sin_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\cos_{q^{-\frac{1}{2}}}(a;z_0;z),
\end{aligned} \tag{2.98}$$

where $\mathcal{D}^+f(x(z)) = Df(x(z))\big|_{z:=zq^{\frac{1}{2}}}$. Hence

$$\begin{aligned}
[\mathcal{D}^+]^2\cos_{q^{-\frac{1}{2}}}(a;z_0;z) &= -a^2\cos_{q^{-\frac{1}{2}}}(a;z_0;z),\\
[\mathcal{D}^+]^2\sin_{q^{-\frac{1}{2}}}(a;z_0;z) &= -a^2\sin_{q^{-\frac{1}{2}}}(a;z_0;z).
\end{aligned} \tag{2.99}$$

Similarly

$$\begin{aligned}
\mathcal{D}\cosh_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\sinh_{q^{-\frac{1}{2}}}(a;z_0;zq^{-\frac{1}{2}});\\
\mathcal{D}^-\cosh_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\sinh_{q^{-\frac{1}{2}}}(a;z_0;z),\\
\mathcal{D}\sinh_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\cosh_{q^{-\frac{1}{2}}}(a;z_0;zq^{-\frac{1}{2}});\\
\mathcal{D}^-\sinh_{q^{-\frac{1}{2}}}(a;z_0;z) &= a\cosh_{q^{-\frac{1}{2}}}(a;z_0;z),
\end{aligned}$$

and consequently

$$[\mathcal{D}^+]^2 \cosh_{q^{-\frac{1}{2}}}(a; z_0; z) = a^2 \cosh_{q^{-\frac{1}{2}}}(a; z_0; z),$$
$$[\mathcal{D}^+]^2 \sinh_{q^{-\frac{1}{2}}}(a; z_0; z) = a^2 \sinh_{q^{-\frac{1}{2}}}(a; z_0; z). \qquad (2.100)$$

From the preceding, one easily verifies that

a)

$$\cos^2_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a; z_0; z) + \sin^2_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a; z_0; z) = 1, \qquad (2.101)$$

b)

$$\cosh^2_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a; z_0; z) - \sinh^2_{q^{\frac{1}{2}}q^{-\frac{1}{2}}}(a; z_0; z) = 1, \qquad (2.102)$$

c)

$$\cos_{q^{\frac{1}{2}}}(a; z_0; z) \cos_{q^{-\frac{1}{2}}}(a; z_0; z) +$$
$$\sin_{q^{\frac{1}{2}}}(a; z_0; z) \sin_{q^{-\frac{1}{2}}}(a; z_0; z) = 1, \qquad (2.103)$$

d)

$$\cosh_{q^{\frac{1}{2}}}(a; z_0; z) \cosh_{q^{-\frac{1}{2}}}(a; z_0; z) -$$
$$\sinh_{q^{\frac{1}{2}}}(a; z_0; z) \sinh_{q^{-\frac{1}{2}}}(a; z_0; z) = 1. \qquad (2.104)$$

2.2 Systems of linear q-Nonuniform difference equations

2.2.1 The general theory

A linear q-nonuniform difference system may be written as

$$\mathcal{D}y(x(z)) = A(x(z))y(x(q^{-\frac{1}{2}}z)) + b(x(z)), \tag{2.105}$$

or equivalently by replacing q by q^{-1},

$$\mathcal{D}\tilde{y}(x(z)) = \tilde{A}(x(z))\tilde{y}(x(q^{\frac{1}{2}}z)) + \tilde{b}(x(z)). \tag{2.106}$$

where y, $\tilde{y}$, b and $\tilde{b}$ are k-vectors, A and $\tilde{A}$, $k \times k$ non singular matrices.

The homogenous part of the equation (2.105), which is

$$\mathcal{D}y(x(z)) = A(x(z))y(x(q^{-\frac{1}{2}}z)) \tag{2.107}$$

may be expressed recursively as

$$y(x(q^{\frac{1}{2}}z)) = P(z)y(x(q^{-\frac{1}{2}}z)) \tag{2.108}$$

where

$$P(z) = I + \tfrac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})A(x(z)) \tag{2.109}$$

admitting a recursive solution

$$y(x(z)) = \left[\prod_{i=0}^{N-1}\left[P(q^{\frac{1}{2}+i}z)\right]^{-1}\right] y(x(q^N z)). \tag{2.110}$$

Next, consider the associated linear matrix systems

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(q^{-\frac{1}{2}}z)), \tag{2.111}$$

and

$$\mathcal{D}\tilde{Y}(x(z)) = \tilde{Y}(x(q^{\frac{1}{2}}z))\tilde{A}(x(z)). \tag{2.112}$$

In recursive forms they read

$$Y(x(q^{\frac{1}{2}}z)) = P(z)Y(x(q^{-\frac{1}{2}}z)) \tag{2.113}$$

and

$$\tilde{Y}(x(q^{\frac{1}{2}}z))\tilde{P}(z) = \tilde{Y}(x(q^{-\frac{1}{2}}z)) \tag{2.114}$$

where

$$P(z) = I + \tfrac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})A(x(z)) \tag{2.115}$$

and

$$\tilde{P}(z) = I - \tfrac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})\tilde{A}(x(z)). \tag{2.116}$$

The solution of (2.111) verifies

$$Y(x(z)) = \left[\prod_{i=0}^{N-1}\left[P(q^{\frac{1}{2}+i}z)\right]^{-1}\right] Y(x(q^N z)) \tag{2.117}$$

while that of (2.112) verifies

$$\tilde{Y}(x(z)) = \tilde{Y}(x(q^N z))\left[\prod_{i=N-1}^{0}\tilde{P}(q^{\frac{1}{2}+i}z)\right]. \tag{2.118}$$

As in the scalar case, we can also define the exponential matrix funtions

$$E_{q^{-\frac{1}{2}}}(P;N;z) \stackrel{\text{def}}{=} \prod_{i=0}^{N-1} [P(q^{\frac{1}{2}+i}z)]^{-1}, \tag{2.119}$$

and

$$E_{q^{\frac{1}{2}}}(\tilde{P};N;z) \stackrel{\text{def}}{=} \prod_{i=N-1}^{0} \tilde{P}(q^{\frac{1}{2}+i}z). \tag{2.120}$$

From (2.115) and (2.116) it is clear that for $\tilde{A}(x(z)) = -A(x(z))$, then $\tilde{P}(x(z)) = P(x(z))$ and multiplying of (2.117) with (2.118) gives $\tilde{Y}(x(z))Y(x(z)) = \tilde{Y}(x(q^N z))Y(x(q^N z))$. This means that multiplying the solutions of (2.111) and (2.112) with $\tilde{A}(x(z)) = -A(x(z))$ gives a constant matrix. In case this matrix is unitary, the solutions are mutually inverse. More generally we have

Theorem 2.3. *If $Y(x(z))$ and $\tilde{Y}(x(z))$ are respective solutions of the associated matrix systems*

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(q^{-\frac{1}{2}}z)) \tag{2.121}$$

$$\mathcal{D}\tilde{Y}(x(z)) = -\tilde{Y}(x(q^{\frac{1}{2}}z))A(x(z)) \tag{2.122}$$

satisfying the conditions

$$\tilde{Y}(x(z_0))Y(x(z_0)) = I \tag{2.123}$$

then these matrix functions are mutually inverse.

Proof. Applying the q-nonuniform derivative to the product

$$\tilde{Y}(x(z))Y(x(z))$$

we get

$$\mathcal{D}(\tilde{Y}Y) = \mathcal{D}\tilde{Y}.Y(x(zq^{-\frac{1}{2}})) + \tilde{Y}(x(zq^{\frac{1}{2}})).\mathcal{D}Y$$
$$= -\tilde{Y}(x(zq^{\frac{1}{2}}))A(x(z))Y(x(zq^{-\frac{1}{2}})) + \tilde{Y}(x(zq^{\frac{1}{2}}))A(x(z))Y(x(q^{-\frac{1}{2}})) = 0. \tag{2.124}$$

This means that $\tilde{Y}(x(z))Y(x(z)) = const$, and the condition (2.123) brings the desired result. □

Corollary 2.4. *The exponential matrix functions (2.119) and (2.120) are mutually inverse, that is*

$$E_{q^{\frac{1}{2}}}(-P; N; z).E_{q^{-\frac{1}{2}}}(P; N; z) = 1.$$

The theorem 2.3 will also be used in the proof of the *q-nonuniform duality theorem* of the next chapter.

Return now to the linear systems (2.105), (2.107). It is easily seen that if a nonsingular matrix $Y_0(x(z))$ solves (2.111), then the general solution of (2.107) reads $y = Y_0 c$ for constant k-vector c. Using the *method of variation of constants*, one finds next, the general solution of the non homogeneous system (2.105):

$$\begin{aligned} y(x(z)) &= \Phi(x(z), x(z_0))[y(x(z_0)) \\ &+ \int_{x(z_0)}^{x(z)} \Phi(x(z_0), x(q^{\frac{1}{2}}z))b(x(z))d_q x(z)], \end{aligned} \tag{2.125}$$

where $\Phi(a, b)$ is given by

$$\Phi(a, b) = Y_0(a)Y_0^{-1}(b), \tag{2.126}$$

and is referred to as the *q-nonuniform transition matrix*. The general solution of the system (2.106) associated to the system (2.105) reads

$$\begin{aligned} y(x(z)) &= \tilde{\Phi}(x(z), x(z_0))[y(x(z_0)) \\ &+ \int_{x(z_0)}^{x(z)} \tilde{\Phi}(x(z_0), x(q^{-\frac{1}{2}}z))b(x(z))d_q x(z)], \end{aligned} \tag{2.127}$$

where $\tilde{\Phi}(a, b)$ is given by

$$\tilde{\Phi}(a, b) = \tilde{Y}_0(a)\tilde{Y}_0^{-1}(b), \tag{2.128}$$

The second section of the next chapter is devoted to the controllability of type (2.105) systems with $b(x(z)) = B(x(z))u(x(z))$, B-a $k \times m$ matrix, and u-a $m \times 1$ control function. As we will see, in controllability theory, the form (2.126) for the solution of (2.105) is the most adequate for the analysis.

2.2.2 Autonomous systems

The treatement of the autonomous systems cases of the equations (2.105) and (2.106) are in general similar to the one done for scalar cases of first paragraph, and we will note reproduice them here.

Instead, we will consider the following case of the q-nonuniform linear first order autonomous system

$$\mathcal{D}y(x(z)) = Ay(x(z)) \tag{2.129}$$

where A-a $k \times k$ matrix and b-a k vector.

Contrarly to the systems (2.105) and (2.106), this system can be solved by the well known method of *Picard approximations*. For that, we consider the following sequence

$$y_0(x(z)) = y_0, \;\; y_k(x(z)) = \int\limits_{x(z_0)}^{x(z)} A(x(z))y_{k-1}(x(z))d(x(z)), \;\; k = 1, 2, ... \tag{2.130}$$

and we can then easily check that the formal series

$$y(x(z)) = \sum_{k=0}^{\infty} y_k(x(z)), \tag{2.131}$$

satisfies the equation (2.129).

In this case

$$y_k(x(z)) = A^k P_k(x(z)) \tag{2.132}$$

where $P_k(x(z))$ are polynomials of degree k in $x(z)$ and defined by

$$P_0(x(z)) = 1, \quad P_k(x(z)) = \int\limits_{x(z_0)}^{x(z)} P_{k-1}(x(z))dx(z), \quad k = 1, 2, \cdots . \tag{2.133}$$

The solution of (2.129) is then the following formal series

$$E_q(A; x(z)) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} A^k P_k(x(z)). \tag{2.134}$$

Here, we have used the first fundamental principle of q-nonuniform calculus as defined in the first chapter and the fact that a q-nonuniform

integral of a polynomial of degre $k-1$ in $x(z)$ gives a polynomial of degre k in $x(z)$. However, the polynomials $P_k(x)z))$ must be understood here as q-nonuniform versions of the powers $\frac{(t-t_0)^k}{k!}$

2.3 q-Nonuniform difference equations of higher order

2.3.1 The general theory

Consider the linear q-nonuniform difference equations of higher order

$$[a_0(x(z))(\mathcal{D}^+)^n + a_1(x(z))(\mathcal{D}^+)^{n-1} + \cdots + a_n(x(z))]y(x(z)) = g(x(z)) \quad (2.135)$$

and its homogenous part

$$[a_0(x(z))(\mathcal{D}^+)^n + a_1(x(z))(\mathcal{D}^+)^{n-1} + \cdots + a_n(x(z)]y(x(z)) = 0. \quad (2.136)$$

The scalar equation (2.135) can be handled as a particular case of the system (2.105). Indeed, by the change of variables

$$\begin{gathered} u_1(x) = y(x); \\ u_2(x) = \mathcal{D}^+ y(x); \ldots; u_k(x) = (\mathcal{D}^+)^{k-1} y(x), \end{gathered} \quad (2.137)$$

simple operations transform (2.135) in type (2.105) system namely

$$\begin{aligned}
\mathcal{D}^+u_1(x) &= u_2(x),\\
\mathcal{D}^+u_2(x) &= u_3(x),\\
&\cdots\\
\mathcal{D}^+u_{k-1}(x) &= u_k(x)\\
\mathcal{D}^+u_k(x) &= -(a_1(x)u_k(x)+\cdots+a_k(x)u_1(x))+g(x) \qquad (2.138)
\end{aligned}$$

In matrices terms, we have

$$\mathcal{D}u(x(z)) = A(x(z))u(x(q^{-\frac{1}{2}}z))+h(x(z)) \qquad (2.139)$$

where $u(x)=(u_1(x),\ldots,u_k(x))^t$,

$$A(x(z)) = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0\\
0 & 0 & 1 & 0 & \ldots & 0\\
. & . & . & . & . & .\\
0 & . & . & . & 0 & 1\\
-a_k(x(zq^{-\frac{1}{2}})), & . & . & . & . & -a_1(x(zq^{-\frac{1}{2}}))
\end{pmatrix} \qquad (2.140)$$

and $h(x(z)) = (0,\ldots,0,g(x(q^{-\frac{1}{2}}z)))^t$. So, from (2.137), it follows that the existence of a unique solution of (2.135) under the initial constraints $y(x_0)=y_0$, $\mathcal{D}^+y(x_0)=y_1$, $\ldots$, $(\mathcal{D}^+)^{k-1}y(x_0)=y_{k-1}$, is equivalent to the existence of a unique solution of (2.139) under the constraints $(u_1(x_0),\ldots,u_k(x_0))^t=(y_0,\ldots,y_{k-1})^t$. As a consequence, the existence of a fundamental system of solutions $y_1(x),\ldots,y_k(x)$ of (2.136) is equivalent to the existence of a fundamental system $(y_1(x),\mathcal{D}^+y_1(x),\ldots,(\mathcal{D}^+)^{k-1}y_1(x))^t$, $\ldots$, $(y_k(x),\mathcal{D}^+y_k(x),\ldots,(\mathcal{D}^+)^{k-1}y_k(x))^t$ of the system

$$\mathcal{D}u(x) = A(x)u(x(zq^{-\frac{1}{2}})), \qquad (2.141)$$

with the fundamental matrix

$$\Phi(x) = \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_k(x) \\ \mathcal{D}^+ y_1(x) & \mathcal{D}^+ y_2(x) & \dots & \mathcal{D}^+ y_k(x) \\ . & . & . & . \\ (\mathcal{D}^+)^{k-1} y_1(x) & (\mathcal{D}^+)^{k-1} y_2(x) & \dots & (\mathcal{D}^+)^{k-1} y_k(x) \end{pmatrix} \tag{2.142}$$

As in differential equations theory, this relation allows us to derive the general solution of (2.135) using the knowledge of the general solution of linear systems. If $y_1(x), \dots, y_k(x)$ is a fundamental system of solutions of the homogenous equation (2.136), corresponding to the fundamental matrix $\Phi(x)$, then according to the general theory of q-nonuniform difference systems, the general solution of (2.139) is found as

$$u(x) = \Phi(x).c(x) \tag{2.143}$$

where $\Phi(x)$ is given by (2.142) and $c(x) = (c_1(x), \dots, c_k(x))^t$ is the solution of the system

$$\Phi(x(zq^{\frac{1}{2}}))\mathcal{D}c(x) = h(x). \tag{2.144}$$

Hence the general solution of (2.135) reads

$$y(x) = u_1(x) = \sum_{i=1}^{k} c_i(x) y_i(x), \tag{2.145}$$

as in differential equation theory.

2.3.2 Nonlinear q-nonuniform difference equations transformable in linear equations

As in the case of first order, some nonlinear q-nonuniform difference equations are transformable into linear ones. This is the case of Riccati type equation [44]

$$\begin{aligned} a_0(x)\mathcal{D}y(x(z)) &= b_0(x)y(x(zq^{\frac{1}{2}}))y(x(zq^{-\frac{1}{2}})) \\ &+ c_0(x)\frac{y(x(zq^{\frac{1}{2}}))+y(x(zq^{-\frac{1}{2}}))}{2}+d_0(x(z)) \end{aligned} \tag{2.146}$$

It is transformable in linear q-nonuniform second order difference equation by first writing it in the homographic form

$$y(x(zq^{\frac{1}{2}})) = \frac{a(z)y(x(zq^{-\frac{1}{2}}))+b(z)}{c(z)y(x(zq^{-\frac{1}{2}}))+d(z)}. \tag{2.147}$$

where $a(z) = \frac{a_0(x(z))}{x(zq^{\frac{1}{2}})-x(zq^{-\frac{1}{2}})} + \frac{c_0(x(z))}{2}$; $d(z) = \frac{a_0(x(z))}{x(zq^{\frac{1}{2}})-x(zq^{-\frac{1}{2}})} - \frac{c_0(x(z))}{2}$; $b(z) = d_0(x(z))$; $c(z) = -b_0(x(z))$ and then make the change of variable

$$u(x(zq^{\frac{1}{2}}))/u(x(zq^{-\frac{1}{2}}) = c(z)y(x(zq^{-\frac{1}{2}})) + d(z). \tag{2.148}$$

The resulting second order linear q-nonuniform difference equation reads

$$\begin{aligned} &[c(z)]u(x(zq^{3/2})) + [-c(z)d(qz) - c(qz)a(z)]u(x(zq^{1/2})) \\ &+[c(qz)a(z)d(z) - c(qz)b(z)c(z)]u(x(zq^{-1/2})) = 0. \end{aligned} \tag{2.149}$$

2.3.3 Linear q-nonuniform difference equations of second order

Consider now the q-nonuniform linear difference equations of second order. It is clear from the solvability of the first order equation that the derivative to be considered here is not $\mathcal{D}$ but the derivatives $\mathcal{D}^+$ and $\mathcal{D}^-$ defined by

$$\begin{aligned}\mathcal{D}^+y(x(z)) &= \frac{f(x(qz))-f(x(z))}{x(qz)-x(z)};\\ \mathcal{D}^-y(x(z)) &= \frac{f(x(z))-f(x(z/q))}{x(z)-x(z/q)}\end{aligned} \tag{2.150}$$

Thus, we study q-nonuniform linear difference equations of second order of the form

$$\begin{aligned}a_0(x(z))[\mathcal{D}^+]^2y(x(z)) &+ a_1(x(z))\mathcal{D}^+y(x(z))\\ &+ a_2(x(z))y(x(z))=0,\end{aligned} \tag{2.151}$$

or these of the form

$$\begin{aligned}h_0(x(z))\mathcal{D}^-\mathcal{D}^+y(x(z))+h_1(x(z))\mathcal{D}^+y(x(z))+h_2(x(z))y(x(z))=0,\\ a_0,h_0\neq 0,\end{aligned} \tag{2.152}$$

when we are interested in the self-adjointness property.

1. **Solvability**

   As in the differential or difference cases, there is no general way of solving type (2.151) or (2.152) in quadratures for general coefficients $h_0(x(z))$, $h_1(x(z))$, $h_2(x(z))$, $a_0(x(z))$, $a_1(x(z))$ and $a_2(x(z))$. However, some particular cases can be solved explicitly.

(i) Consider the equation (2.151) and suppose that one of its two independent solutions, say $y_1(x(z))$, is known. So the other solution, say $y_2(x(z))$, can be found as in differential or difference calculus using the *q-nonuniform version of the Liouville formula*. We find the latter by comparing equation (2.151) with the following

$$\det\begin{pmatrix} y(x(z)) & y_1(x(z)) & y_2(x(z)) \\ \mathcal{D}^+y(x(z)) & \mathcal{D}^+y_1(x(z)) & \mathcal{D}^+y_2(x(z)) \\ (\mathcal{D}^+)^2y(x(z)) & (\mathcal{D}^+)^2y_1(x(z)) & (\mathcal{D}^+)^2y_2(x(z)) \end{pmatrix} = 0, \quad (2.153)$$

and simple computations show that we have

$$\mathcal{D}^+W(y_1,y_2) = \quad (2.154)$$
$$[-\tilde{a}_1(x(z)) + \tilde{a}_2(x(z))(x(qz) - x(z))]W(y_1,y_2),$$

where $W(u,v) = u\mathcal{D}^+v - v\mathcal{D}^+u$ is the Vronskian, and $\tilde{a}_1 = a_1/a_0, \tilde{a}_2 = a_2/a_0$.

Equation (2.154) is equivalent to the first order q-nonuniform equation for $W(x(z)) = W(y_1,y_2)(x(z))$

$$\mathcal{D}W(x(z)) = a(x(z))W(x(zq^{-\frac{1}{2}})), \quad (2.155)$$

where $a(x(z)) = -\tilde{a}_1(x(zq^{-\frac{1}{2}})) + \tilde{a}_2(x(zq^{-\frac{1}{2}}))(x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))$, whose solution gives the required q-nonuniform Liouville formula:

$$W(x(z)) = \{ \begin{matrix} \prod_{t=qz_0}^{z} p(tq^{-\frac{1}{2}})W(x(z_0)), & z = z_0.q^{k_z} \\ (\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}}))^{-1}W(x(z_0)), & z = z_0.q^{-k_z}, \end{matrix} \quad (2.156)$$

where $p(z)$ is given as in (2.4), or more compactly

$$W(x(z)) = W(x(z_0))\tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_qz}. \quad (2.157)$$

(ii) The next example is when the *coefficients* in equation (2.151) are *constant*. We can solve it by first factorizing it as

$$[(\mathcal{D}^+ - k_2)(\mathcal{D}^+ - k_1)]y(x(z)) = 0 \tag{2.158}$$

where k_1, k_2 are the roots of the characteristic algebraic equation

$$a_0k^2 + a_1k + a_2 = 0 \tag{2.159}$$

and then solve the two resulting first order equations. If $k_1 \neq k_2$, one obtains the two required independent solutions: $y_1(x(z)) = \tilde{e}_q^{k_1x(z)}$ and $y_2(x(z)) = \tilde{e}_q^{k_2x(z)}$. For example, the trigonometric functions $\widetilde{\cos}_q x(z) = \frac{\tilde{e}_q^{ix(z)}+\tilde{e}_q^{-ix(z)}}{2}$, $\widetilde{\sin}_q x(z) = \frac{\tilde{e}_q^{ix(z)}-\tilde{e}_q^{-ix(z)}}{2i}$ and the hyperbolic ones $\widetilde{\cosh}_q x(z) = \frac{\tilde{e}_q^{x(z)}+\tilde{e}_q^{-x(z)}}{2}$, $\widetilde{\sinh}_q x(z) = \frac{\tilde{e}_q^{x(z)}-\tilde{e}_q^{-x(z)}}{2}$ are solutions of the equations

$$(\mathcal{D}^+)^2y(x(z)) + y(x(z)) = 0 \tag{2.160}$$

and

$$(\mathcal{D}^+)^2y(x(z)) - y(x(z)) = 0 \tag{2.161}$$

respectively. On the other hand, if (2.159) admits a double root, then one uses the Liouville formula in (2.157) to find the second independent solution.

(iii) A particular case of (2.152) is well known: If

$$h_0(x(z)) = \frac{x(qz) - x(z)}{x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}})}[\sigma(x(z)) - \frac{1}{2}\tau(x(z))(x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))] \tag{2.162}$$

and

$$h_1(x(z)) = \tau(x(z));\ h_2(x(z)) = const, \tag{2.163}$$

where σ and τ are polynomials of degree ≤ 2 and 1 respectively, then (2.152) admits a system of polynomial solutions, the famous Askey-Wilson polynomials [9]. The equation will be solved in section 4.1, using the factorization method studied in [11].

2. **Orthogonality**

Consider the eigenvalue equation

$$a(x(z))\mathcal{D}^-\mathcal{D}^+y(x(z)) + b(x(z))\mathcal{D}^+y(x(z)) = \lambda y(x(z)) \quad (2.164)$$

We can write it under the self-adjoint form

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+y(x(z))] = \lambda\varrho(x(z))y(x(z)). \quad (2.165)$$

where

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))] = \varrho(x(z))b(x(z)). \quad (2.166)$$

Suppose that we also have

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+\tilde{y}(x(z))] = \tilde{\lambda}\varrho(x(z))\tilde{y}(x(z)), \quad (2.167)$$

for distinct $\tilde{\lambda}$ and λ. Next, multiply equation (2.165) by $\tilde{y}((z))$ and equation (2.167) by $y((z))$ and subtract member by member. We get

$$\begin{aligned}&\tilde{y}(x(z))\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+y(x(z))]\\ &-y(x(z))\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+\tilde{y}(x(z))]\\ &\quad = (\lambda-\tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)). \quad (2.168)\end{aligned}$$

Next, by the easily verified identity

$$\mathcal{D}^{-}\left[p(u\mathcal{D}^{+}v - v\mathcal{D}^{+}u)\right] = u\mathcal{D}^{-}(p\mathcal{D}^{+}v) - v\mathcal{D}^{-}(p\mathcal{D}^{+}u) \quad (2.169)$$

we obtain

$$\begin{aligned}&\mathcal{D}^{-}\left[a(x(qz))\varrho(x(qz))[\tilde{y}\mathcal{D}^{+}y - y\mathcal{D}^{+}\tilde{y}]\right]\\ &\quad = (\lambda - \tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)),\end{aligned} \quad (2.170)$$

or

$$\begin{aligned}&\mathcal{D}^{-}\left[a(x(qz))\varrho(x(qz))W(\tilde{y},y)\right] =\\ &(\lambda - \tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)).\end{aligned} \quad (2.171)$$

Integrating both sides of the equality from say, $x(z_0)$ to $x(z_1)$, noting that $(\mathcal{D}^{-}F)(z) = (\mathcal{D}F)(q^{-\frac{1}{2}}z)$, we get

$$\begin{aligned}&(\lambda - \tilde{\lambda})\int_{x(z_0)}^{x(z_1)} \varrho(x(z))y(x(z))\tilde{y}(x(z))d_qx(z)\\ &= \int_{x(z_0)}^{x(z_1)} \mathcal{D}^{-}\left[a(x(qz))\varrho(x(qz))W(\tilde{y},y)\right]d_q(x(z))\\ &= a(x(qz))\varrho(x(qz))W(\tilde{y},y)\,|_{x(z_0q^{-\frac{1}{2}})}^{x(z_1q^{-\frac{1}{2}})}.\end{aligned} \quad (2.172)$$

The latter gives the *orthogonality condition* for two eigenfunctions $y(x(z))$ and $\tilde{y}(x(z))$ of (2.164) corresponding to distinct eigenvalues λ and $\tilde{\lambda}$, on the interval $[x(z_0), x(z_1)]$, with a q-nonuniform discrete weight $\varrho(x(z))$ given by (2.166):

$$a(x(qz))\varrho(x(qz))W(\tilde{y}(x(z)), y(x(z)))\,|_{x(z_0q^{-\frac{1}{2}})}^{x(z_1q^{-\frac{1}{2}})} = 0. \quad (2.173)$$

Chapter 3

q-Nonuniform difference control theory

3.1 q-Nonuniform difference variational calculus

This chapter concerns the generalization of the variational calculus. The difference variational calculus was proposed in [26], while in [12], the q-uniform difference variational calculus was studied. Here, we consider the q-nonuniform difference variational calculus studied in [13]. More precisely, we discuss q-nonuniform difference versions of the basic concepts of variational calculus such as the Euler-Lagrange equation, the isoperimetric, Lagrange and optimal control problems. Also, some interconnections between the q-nonuniform difference Euler-Lagrange equation, the q-nonuniform difference Hamilton and Hamilton-Pontryaguin systems are discussed.

3.1.1 q-Nonuniform difference Euler-Lagrange equation

We consider the following functional given as a q-nonuniform integral:

$$J(y(x(z))) = \int_a^b F[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))] \tag{3.1}$$

where $\gamma(z)$ is given by

$$\gamma(z) = x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z) \tag{3.2}$$

and $a = x(q^{\beta+\frac{1}{2}})$, $b = x(q^{\alpha-\frac{1}{2}})$ with the supposition that $\beta \geq \alpha$. In (3.1), F is a differentiable function with respect to all its arguments. The function y belongs to the variety E' of functions satisfying boundary constraints

$$y(q^{\alpha-\frac{1}{2}}) = y(q^{\beta+\frac{1}{2}}) = c, \tag{3.3}$$

in the linear space E of functions $f(x(z))$ defined and bounded together with $\mathcal{D}f(x(z))$, on the set

$$L = \{q^{\alpha-\frac{1}{2}+\frac{i}{2}},\ i = 0, 1, \ldots, 2(1+\beta-\alpha)\} \tag{3.4}$$

and equipped with the norm

$$\|f\| = \max(\sup_{z\in L} |f(x(z))|,\ \sup_{z\in L} |\mathcal{D}f(x(z))|). \tag{3.5}$$

The extremum problem consists then in finding the extremals for the functional (3.1) under the constraints (3.3). As F is a differentiable function with respect to all its arguments, we can calculate the first

variation of the functional:

$$\delta J(y(x(z)),h(x(z))) = \frac{d}{dt}J(y(x)+th(x))|_{t=0}$$
$$= \frac{d}{dt}\left[\int_a^b F[x(z),y(x(q^{-\frac{1}{2}}z))+th(x(q^{-\frac{1}{2}}z)),\mathcal{D}y(x(z))+t\mathcal{D}h(x(z))]d_qx(z)\right]_{t=0}.$$

Hence

$$\delta J(y(x(z)),h(x(z))) =$$
$$\sum_{z=q^\alpha}^{q^\beta}\gamma(z)\{F_{v_0}[x(z),y(x(q^{-\frac{1}{2}}z)),\mathcal{D}y(x(z))]h(x(q^{-\frac{1}{2}}z))$$
$$+F_{v_1}[x(z),y(x(q^{-\frac{1}{2}}z)),\mathcal{D}y(x(z))]\mathcal{D}h(x(z))\} \tag{3.6}$$

where $F_{v_i}=\frac{\partial F}{\partial v_i}$ means the derivative of F with respect to its $(i+2)^{th}$ argument, $i=0,1$. As $y+th$ belongs also to E', it follows from (3.3) that

$$h(q^{\alpha-\frac{1}{2}})=h(q^{\beta+\frac{1}{2}})=0. \tag{3.7}$$

Using (1.24) and (3.7), one transforms (3.6) in

$$\delta J(y(x(z)),h(x(z))) =$$
$$\sum_{z=q^\alpha}^{q^\beta}\gamma(z)\{F_{v_0}[x(z),y(x(q^{-\frac{1}{2}}z)),\mathcal{D}y(x(z))]$$
$$-\mathcal{D}[F_{v_1}[x(zq^{-\frac{1}{2}}),y(x(q^{-1}z)),\mathcal{D}y(x(q^{-\frac{1}{2}}z))]]\}h(x(q^{-\frac{1}{2}}z)). \tag{3.8}$$

To obtain the Euler-Lagrange equation form (3.8), we need the following q-nonuniform difference version of the "fundamental lemma of variational calculus"

Lemma 3.1. *Suppose that for a given function $f(z)$, one has*

$$\sum_{q^\alpha}^{q^\beta}\gamma(z)f(z)p(z)=0 \tag{3.9}$$

for any function $p(z)$ belonging to the space E, then $f(z) \equiv 0$.

Proof. For various functions $p_i(z)$, the equation (3.9) gives a system of equations that one can write in matrix (may be infinite dimensional) form as $Ay = 0$ where $A_{ij} = \gamma(q^{\alpha+j})p_i(q^{\alpha+j})$ and $y_j = f(q^{\alpha+j})$, $i, j = 0, \ldots, \beta - \alpha$. To obtain $f(z) \equiv 0$, it suffices to choose the $p_i(q^{\alpha+j})$ so that the matrix A be invertible, which proves the lemma. □

Applying the lemma to the equation (3.8), one obtains

$$\begin{aligned} &F_{v_0}[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))] \\ &-\mathcal{D}[F_{v_1}[x(zq^{-\frac{1}{2}}), y(x(q^{-1}z)), \mathcal{D}y(x(q^{-\frac{1}{2}}z))]] = 0, \end{aligned} \tag{3.10}$$

which is the *Euler-Lagrange equation* giving the necessary condition for the q-nounuform difference extremum problem. It is a second order q-nonuniform difference equation, which in principle is solved uniquely under the boundary constraints (3.3).

Remark 3.2. If the function under the sign of integration F is given by $F = F(x, y_1, \ldots, y_n, \mathcal{D}y_1, \ldots, \mathcal{D}y_n)$, so the extremum necessary condition is given by n equations similar to (3.10), one equation for each variable, the other variables being supposed fixed.

Remark 3.3. If the function under the sign of integration F is given by $F = F(x, y, \mathcal{D}y, \ldots, \mathcal{D}^n y)$, so the change of variables $y_1 = y, y_2 = \mathcal{D}y, \ldots, y_n = \mathcal{D}^{n-1}y$ leads to the case of remark 1, with additional constraints: $\mathcal{D}y_1 = y_2, \mathcal{D}y_2 = y_3 \ldots, \mathcal{D}y_{n-1} = y_n$. This gives a particular case of the "Lagrange problem" which will be discussed in the next section.

3.1.2 Applications

1. **q-Nonuniform difference isoperimetric problem**

The problem. Consider the q-nonuniform integration functional

$$J_0(y(x(z))) = \int_a^b F_0[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F_0[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))] \qquad (3.11)$$

defined in E'. Let next be given a set of other functionals

$$J_i(y(x(z))) = \int_a^b F_i[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F_i[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))], \qquad (3.12)$$

defined also in E' and consider the equations

$$J_i(y(x)) = c_i, \ i = 1, \ldots, m. \qquad (3.13)$$

The isoperimetric problem consists in finding extremals of the functional $J_0(y)$, among all the functions belonging in E' and satisfying (3.13).

The solution. The settled isoperimetric problem can be solved in a more general setting by the following theorem (see for ex [36]).

Theorem 3.4. *Suppose that is given a set of functionals $J_i(y), i = 0, 1, \ldots, m$ defined on a variety E' of a linear normed space E and admitting on E' the first variation $\delta J_i(y, h)$ with $\delta J_i(y, h)$, $i = 1, \ldots, m$ linearly independent functionals. Suppose next that*

y_0 is an extremal of $J_0(y)$ under the constraints $J_i(y) = c_i$, $i = 1, \ldots, m$ and $\delta J_i(y_0, h) \not\equiv 0$, $i = 1, \ldots, m$. In that case, y_0 is an ordinary extremal for the functional $J^*(y) = J_0(y) + \sum_{i=1}^{m} \lambda_i J_i(y)$, where the λ_i are some constants.

Proof. Applied to our functionals (3.11), (3.12) and constraints (3.13), the theorem implies that if the functions under the signs of integration F_0 and F_i are differentiable with respect to all its arguments (this is sufficient for the functionals to have the first variation), and the variational derivatives (i.e. the function in the lhs of the corresponding Euler-Lagrange equation) of the functionals (3.12) are linearly independent (this is sufficient for the first variations to be so), then the extremals of $J_0(y)$ under the constraints (3.13) are included in the union of the set of solutions of the equations $\delta J_i(y, h) = 0$, $i = 1, \ldots, m$, and that of $\delta J^*(y, h) = 0$, where $J^*(y) = J_0(y) + \sum_{i=1}^{m} \lambda_i J_i(y)$. □

Example. Suppose it required to find the extremum of the q-nonuniform integration functional

$$J_0(y(x(z))) = \int_a^b [\tfrac{1}{2}(\mathcal{D}y(x(z)))^2 - a(q + q^2 + 2q^{\frac{3}{2}})y(x(q^{-\frac{1}{2}}z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z)[\tfrac{1}{2}(\mathcal{D}y(x(z)))^2 - a(q + q^2 + 2q^{\frac{3}{2}})y(x(q^{-\frac{1}{2}}z))] \qquad (3.14)$$

under the constraint

$$J_1(y(x(z))) = 4aq^{\frac{3}{2}} \int_a^b [x^2(q^{-\frac{3}{4}}z)y(x(q^{-\frac{1}{2}}z))]d_q x(z)$$
$$= 4aq^{\frac{3}{2}} \sum_{z=q^\alpha}^{q^\beta} \gamma(z)[x^2(q^{-\frac{3}{4}}z)y(x(q^{-\frac{1}{2}}z))] = c_1. \qquad (3.15)$$

According to the theorem 3.4, this is equivalent to the problem of finding the ordinary extremum for the functional

$$J^*(y(x(z))) = \int_a^b F^*[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F^*[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))] \qquad (3.16)$$

where

$$F^* = \tfrac{1}{2}(\mathcal{D}y(x(z)))^2$$
$$+[4\lambda a q^{\frac{3}{2}} x^2(q^{-\frac{3}{4}}z) - a(q + q^2 + 2q^{\frac{3}{2}})]y(x(q^{-\frac{1}{2}}z)). \qquad (3.17)$$

The Euler-Lagrange equation for this problem is

$$\mathcal{D}[\mathcal{D}y(q^{-\frac{1}{2}}z)] = 4\lambda a q^{\frac{3}{2}} x^2(q^{-\frac{3}{4}}z) - a(q + q^2 + 2q^{\frac{3}{2}}). \qquad (3.18)$$

Its solution reads

$$y(x(z)) = \tfrac{4aq^4}{(q^3+q^2+q+1)(q^2+q+1)} x^4((z)) - \tfrac{qa(q^5+q^4+4q^3+4q^2+q+1)}{(q^2+1)(q^2+q+1)} x^2((z))$$
$$+\tfrac{q(q^4+q^3+3q^2+q+1)}{2(q^3+q^2+q+1)} + p_1(x(z)),\ \lambda = 1, \qquad (3.19)$$

where $p_1(x(z))$ is any first degree polynomial in $x(z)$.

2. **q-Nonuniform difference Lagrange problem**

The Problem. Let now be given a q-nonuniform integration functional

$$J_0(\bar{y}(x)) =$$
$$\int_a^b F_0[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \mathcal{D}\bar{y}(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F_0[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \mathcal{D}\bar{y}(x(z))] \qquad (3.20)$$

defined in E'^n. Here $\bar{y}(x) = (y_1(x), \ldots, y_n(x))$. Let moreover be given a set of q-nonuniform difference equations

$$\phi_i[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \mathcal{D}\bar{y}(x(z))] = 0, \ i = 1, \ldots, m < n \quad (3.21)$$

The Lagrange problem consists in finding extremals of the functional (3.20) under the constraints (3.21).

The solution. The Lagrange problem can be reduced to the isoperimetric one by transforming (3.21) in type (3.13) constraints. For that, we multiply the both sides of (3.21) by arbitrary functions $\lambda_i(x)$, and then take the integral on the q-nonuniform lattice from a to b. We obtain new constraints

$$J_i(\bar{y}(x)) = \int_a^b \lambda_i(x)\phi_i[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \mathcal{D}\bar{y}(x(z))]d_q x(z) = 0, \quad i = 1, \ldots m. \quad (3.22)$$

Under the conditions of theorem 3.4, the solutions $(y_1(x), \ldots, y_n(x))$ of the isoperimetric problem (3.20), (3.22) satisfy the Euler-Lagrange equation for the functional

$$\hat{J}(\bar{y}) = J_0(\bar{y}) + \sum_{i=1}^{m} \hat{\lambda}_i J_i(\bar{y}), \quad (3.23)$$

for some constants $\hat{\lambda}_i$. But since clearly from (3.21) follows (3.22), the solutions of the Lagrange problem (3.20), (3.21) satisfy as well the Euler-Lagrange equation for the same functional (3.23).

Example. Suppose now that it is required to find the extremum of the functional

$$J_0(x,y,u) = \tfrac{1}{2}\int_a^b [u^2(t(z)) - x^2(t(q^{-\frac{1}{2}}z))]d_q t(z)$$
$$= \tfrac{1}{2}\sum_{z=q^\alpha}^{q^\beta} \gamma(z)[u^2(t(z)) - x^2(t(q^{-\frac{1}{2}}z))], \tag{3.24}$$

under the constraints

$$\mathcal{D}x(t(z)) = y(t(q^{-\frac{1}{2}}z));\ \mathcal{D}y(t(z)) = u(t(z)). \tag{3.25}$$

This is a Lagrange type problem hence it is equivalent to the problem of finding an ordinary extremum for the functional

$$J^*(x,y,u,\lambda_1,\lambda_2) =$$
$$\int_a^b F^*[t(z), x(t(q^{-\frac{1}{2}}z)), y(t(q^{-\frac{1}{2}}z)), u(t(z)), \mathcal{D}x(t(z)), \mathcal{D}y(t(z))]d_q t(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z)F^*[t(z), x(t(q^{-\frac{1}{2}}z)), y(t(q^{-\frac{1}{2}}z)), u(t(z)), \mathcal{D}x(t(z)), \mathcal{D}y(t(z))], \tag{3.26}$$

where

$$F^* = \tfrac{1}{2}(u^2(t(z)) - x^2(t(q^{-\frac{1}{2}}z))) + \lambda_1(t)(\mathcal{D}x(t(z))$$
$$-y(t(q^{-\frac{1}{2}}z))) + \lambda_2(t)(\mathcal{D}y(t(z)) - u(t(z))). \tag{3.27}$$

The Euler-Lagrange equation for this problem reads

$$x(t(z))) = \mathcal{D}^+\mathcal{D}^-\mathcal{D}^-\mathcal{D}^+x(t(z))). \tag{3.28}$$

Searching the solution under the form

$$x(t(z)) = \sum_{j=0}^{\infty}(a_j z^j + a_{-j}z^{-j}), \tag{3.29}$$

one finds the following recurrence relations for the coefficients

$$a_j = \overset{+}{\int_a}\overset{-}{\int_a}\overset{-}{\int_a}\overset{+}{\int_a}(a_j) \tag{3.30}$$

where the applications $\overset{\pm}{\int_a}$:

$$\overset{\pm}{\int_a} : (a_{\pm(j-1)}, a_{\pm j}, a_{\pm(j+1)}) \longrightarrow a^{\pm}_{\pm j} \tag{3.31}$$

are given by

$$\begin{gathered} a^{\pm}_j = \tfrac{\gamma_0 q^{\frac{j}{2}}}{1-q^j}(a_{j-1}q^{\mp\frac{j-1}{2}} - a_{j+1}q^{\mp\frac{j+1}{2}}),\ \gamma_0 = \tfrac{q^{-\frac{1}{2}}-q^{\frac{1}{2}}}{2} \\ a^{\pm}_{-j} = \tfrac{\gamma_0 q^{\frac{j}{2}}}{1-q^j}(a_{-j+1}q^{\pm\frac{j-1}{2}} - a_{-j-1}q^{\pm\frac{j+1}{2}}),\ j = 2, \dots \\ a^{\pm}_1 = \gamma_0(2a_0 - a_2 q^{\mp});\ a^{\pm}_{-1} = \gamma_0(2a_0 - a_{-2}q^{\pm});\ a^{\pm}_0 = cte. \end{gathered} \tag{3.32}$$

such that the applications

$$\overset{\pm}{\int} : \sum_{j=0}^{\infty}(a_j z^j + a_{-j}z^{-j}) \longrightarrow \sum_{j=0}^{\infty}(a^{\pm}_j z^j + a^{\pm}_{-j}z^{-j}) \quad , \tag{3.33}$$

are the inverses of $\mathcal{D}^{\pm}$. Additional constraints to (3.29) are obtained by the fact that the applications $\overset{+}{\int}$ and $\overset{-}{\int}$ are defined on series $\sum_{j=0}^{\infty}(b_j z^j + b_{-j}z^{-j})$ for which $b_1 = qb_{-1}$ and $b_{-1} = qb_1$, respectively.

3. q-Nonuniform difference optimal control problem

The problem. Consider now the q-nonuniform integration functional

$$J(\bar{y}(x),\bar{u}(x)) = \int_a^b f^0[x(z),\bar{y}(x(q^{-\frac{1}{2}}z)),\bar{u}(x(z))]d_qx(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) f^0[x(z),\bar{y}(x(q^{-\frac{1}{2}}z)),\bar{u}(x(z))] \tag{3.34}$$

where $\bar{y}(x) = (y_1(x),\dots,y_n(x))$ and $\bar{u}(x) = (u_1(x),\dots,u_m(x))$, $m \leq n$. The functional is defined on E'^n union the set of admissible (that is which values belong to a fixed set U in $\mathbf{R}^n$) functions $\bar{u}(x)$, where E'^n is the subset of E^n which elements satisfy the boundary constraints

$$\bar{y}(q^{\alpha-\frac{1}{2}}) = \bar{y}(q^{\beta+\frac{1}{2}}) = C. \tag{3.35}$$

Consider then the q-nonuniform difference equations

$$\mathcal{D}y_i(x(z)) = f_i[x(z),\bar{y}(x(q^{-\frac{1}{2}}z)),\bar{u}(x(z))],\ i = 1,\dots,n. \tag{3.36}$$

The optimal control problem consists in finding among all admissible vector functions $\bar{u}(x)$, that for which the corresponding solution of (3.35), (3.36) is an extremal of the functional (3.34). The functions $\bar{y}(x)$ and $\bar{u}(x)$ are said to constitute an *optimal process* and are called *optimal trajectory* and *optimal control* respectively.

The solution. To solve the optimal control problem, we consider it as a $n+m$ dimensional Lagrange problem: Find $n+m$

functions $(y_1(x), \ldots, y_n(x))$ and $(u_1(x), \ldots, u_m(x))$ that are extremals for (3.34) under the conditions (3.35) and

$$\phi_i(x, \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))) = 0 \tag{3.37}$$

where

$$\begin{aligned} &\phi_i(x, \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))) \\ &= \mathcal{D}y_i(x(z)) - f_i[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))],\ i = 1, \ldots, n. \end{aligned} \tag{3.38}$$

According to the discussions done in the preceding subsection, the solutions of such an extremum problem satisfy necessarily the Euler-Lagrange system of the functional

$$\begin{aligned} J^*(\bar{y}(x), \bar{u}(x)) &= \int_a^b F^*[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))] d_q x(z) \\ &= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F^*[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))] \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} F^*[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))] &= f^0[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))] \\ &+ \sum_{i=1}^{n} \psi_i(x)[\mathcal{D}y_i(x) - f_i[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(q^{-\frac{1}{2}}z))]] \end{aligned} \tag{3.40}$$

The corresponding Euler-Lagrange system is then

$$f^0_{v_j} - \sum_{i=1}^{n} \psi_i(x) f_{i_{v_j}} - \mathcal{D}[\psi_i(x(q^{-\frac{1}{2}}z))] = 0,\ j = 1, \ldots, n \tag{3.41}$$

$$f^0_{w_j} - \sum_{i=1}^{n} \psi_i(x) f_{i_{w_j}} = 0,\ j = 1, \ldots, m. \tag{3.42}$$

Here f^0_i and f_i have as arguments $x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(q^{-\frac{1}{2}}z))$ and g_{v_j} and g_{w_j} mean the partial derivatives of g with respect

to its $(j+1)^{th}$ and $(n+j+1)^{th}$ arguments respectively. Setting

$$H = -f^0[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))] + \sum_{i=1}^{n} \psi_i(x)[f_i[x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(z))]], \tag{3.43}$$

so, (3.36), (3.41) and (3.42) give respectively

$$\mathcal{D}y_i(x) = H_{\psi_i} \tag{3.44}$$

$$\mathcal{D}[\psi_i(x(q^{-\frac{1}{2}}z))] = -H_{y_i},\ j = 1, \ldots, n \tag{3.45}$$

and

$$H_{u_i} = 0,\ j = 1, \ldots, m. \tag{3.46}$$

Thus, the necessary condition for the optimal control problem is given by (3.46), provided is solved the system (3.44)-(3.45).

Due to similarities with the continuous case [29], one can refer to H and (3.44)-(3.45) as *q-nonuniform Hamilton-Pontryaguin function* and *system* respectively.

Example. *q-Nonuniform linear quadratic problem*:

The problem now is that of finding a control function $u(x)$ such that the corresponding solution to the boundary value problem

$$\mathcal{D}y = -ay(x(zq^{-\frac{1}{2}}z)) + u(x(z)),\ a > 0$$
$$y(q^{\alpha-\frac{1}{2}}) = y(q^{\beta+\frac{1}{2}}) \tag{3.47}$$

is an extremal for the functional (*quadratic cost functional on q-nonuniform lattices*)

$$J(y,u) = \tfrac{1}{2}\int_a^b [u^2(x(z)) + y^2(x(q^{-\frac{1}{2}}z))]d_q x(z)$$
$$= \tfrac{1}{2}\sum_{z=q^\alpha}^{q^\beta} \gamma(z)[u^2(x(z)) + y^2(x(q^{-\frac{1}{2}}z))]. \qquad (3.48)$$

The problem is of optimal control type. The Hamilton-Pontryaguin function and system are respectively

$$\begin{aligned} H &= -\frac{1}{2}[y^2(x(q^{-\frac{1}{2}}z)) + u^2(x(z))] \\ &+ \psi(x(z))[-ay(x(zq^{-\frac{1}{2}}z)) + u(x(z))] \end{aligned} \qquad (3.49)$$

and

$$\mathcal{D}y = -ay(x(zq^{-\frac{1}{2}}z)) + u(x(z)),$$
$$\mathcal{D}[\psi(x(q^{-\frac{1}{2}}z))] = a\psi(x(z)) + y(x(zq^{-\frac{1}{2}}))$$
$$\psi = u. \qquad (3.50)$$

The equation for $y(x(z))$ then becomes

$$\mathcal{D}[\lceil y(x(q^{-\frac{1}{2}}z)) + ay(x(q^{-1}z))] =$$
$$a\mathcal{D}y + (a^2+1)y(x(q^{-\frac{1}{2}}z)). \qquad (3.51)$$

Searching the solution $y(x(z))$ under a series of the form (3.29), so the recurrence relations satisfied by the coefficients are given by

$$y = \overset{-}{\int} y - a\overset{+}{\int} y + (a^2+1)\overset{-}{\int}\overset{+}{\int} y \qquad (3.52)$$

where the applications $\overset{\pm}{\int}$ are defined in (3.32)-(3.33).

3.1.3 Interconnection between the q-nonuniform difference variational calculus, optimal control and Hamilton system.

Consider now the case of pure q-nonuniform variational calculus that is the control function and the control system are not present explicitly: Find extremals of the functional

$$J(y(x(z))) = \int_a^b F[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]d_q x(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))] \tag{3.53}$$

defined in E'. Our objective is to show the following proposition

Proposition 3.5. *On q-nonuniform lattices, are equivalent: the Euler-Lagrange equation, the Hamilton and the Hamilton-Pontryaguin systems.*

Proof. We show this in three steps:
a) We first show how to obtain the Hamilton system from the Euler-Lagrange equation. For the functional in (3.53), the Euler-Lagrange equation reads

$$F_{v_0}[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))]$$
$$-\mathcal{D}[F_{v_1}[x(zq^{-\frac{1}{2}}), y(x(q^{-1}z)), \mathcal{D}y(x(q^{-\frac{1}{2}}z))]] = 0. \tag{3.54}$$

Letting

$$\psi(x) = F_{v_1}[x(z), y(x(q^{-\frac{1}{2}}z)), \mathcal{D}y(x(z))], \tag{3.55}$$

and

$$H = -F + \psi(x)\mathcal{D}y, \tag{3.56}$$

then we get from (3.54),(3.55) and (3.56) the Hamilton system

$$\begin{aligned}\mathcal{D}y(x(z)) &= H_{\psi}[y(x(q^{-\frac{1}{2}}z)), \psi(x(z)), \mathcal{D}y(x(z))] \\ \mathcal{D}[\psi(x(q^{-\frac{1}{2}}z))] &= -H_{y}[y(x(q^{-\frac{1}{2}}z)), \psi(x(z)), \mathcal{D}y(x(z))] \end{aligned} \quad (3.57)$$

b) To get the Hamilton-Pontryaguin system from the Hamilton system (3.57), it suffices to suppose $u(x(z)) = \mathcal{D}y(x(z))$ to be the control equation for the given initial non controlled extremum problem. In that case, (3.57) gives

$$\begin{aligned}\mathcal{D}y(x(z)) &= H_{\psi}[y(x(q^{-\frac{1}{2}}z)), \psi(x(z)), u(x(z))] \\ \mathcal{D}[\psi(x(q^{-\frac{1}{2}}z))] &= -H_{y}[y(x(q^{-\frac{1}{2}}z)), \psi(x(z)), u(x(z))] \end{aligned} \quad (3.58)$$

with

$$H = -F[y(x(q^{-\frac{1}{2}}z)), u(x)] + \psi(x(z))u(x(z)), \quad (3.59)$$

the Hamilton-Pontryaguin function, and from (3.55) we get the third equation in (3.36):

$$H_u = 0. \quad (3.60)$$

c) We finally show how to obtain the Euler-Lagrange equation (3.54) from the Hamilton-Pontryaguin system (3.58), (3.59) and (3.60). From (3.59) and (3.60), we have

$$\begin{aligned}\psi(x(z)) = F_{v_1}[y(x(q^{-\frac{1}{2}}z)), u(x(z))] = \\ F_1[y(x(q^{-\frac{1}{2}}z)), D_q y(x(z))], \end{aligned} \quad (3.61)$$

while from (3.58) we get

$$\mathcal{D}[\psi(x(q^{-\frac{1}{2}}z))]$$
$$= F_{v_0}[y(x(q^{-\frac{1}{2}}z)), u(x(z))] = F_{v_0}[y(x(q^{-\frac{1}{2}}z)), D_q y(x(z))]. \quad (3.62)$$

Finally, (3.61) and (3.62) give the Euler-Lagrange equation (3.54), which proves the proposition. □

3.2 q-Nonuniform difference linear control systems

The linear control systems theory consists in the study of controllability of linear systems, that is, a set of well defined interconnected objects which interactions can be mathematically modeled by linear systems of divided difference functional equations. Thus a q-nonuniform difference linear control system can be modeled as [16]

$$\mathcal{D}y(x(z)) = A(x(z))y(x(zq^{-\frac{1}{2}})) + B(x(z))u(x(z)) \quad (3.63)$$

where y is a k-vector, A a $k \times k$-matrix, B a $k \times m$-matrix, and u, a m-vector. The vector y stands for the *state variable* of the system, describing the state of the system at a given time s ($z = q^s$), while u stands for the input or the external force constraining the system that is the resulting trajectory to adopt a predetermined behavior. Thus, u controls the system, which is why we speak of control systems. The matrices A and B are intrinsic characterizations or descriptions of the system. In (3.63), the state of the system is described by k variables and the external forces act with m inputs.

In practice, it is often difficult, even impossible, to determine the state of a system itself because it is generally characterized by very numerous variables. Instead, one observes the output of the system $v(x)$, characterized by a small number of variables. Hence, a mathematical model more suitable than (3.63) for the study of the systems controllability reads

$$\begin{aligned} \mathcal{D}y(x(z)) &= A(x(z))y(x(zq^{-\frac{1}{2}})) + B(x(z))u(x(z)) \\ v(x(z)) &= C(x(z))y(x(zq^{\frac{1}{2}})). \end{aligned} \tag{3.64}$$

with C, a $r \times k$-matrix and v, a r-vector, $r < k$.

3.2.1 Controllability

There are many versions of definition of the concept of controllability in mathematical control theory: The controllability of the state, controllability of the output, controllability at the origin, complete controllability and so on. The following definition adopted in this work, consists in the *complete controllability* of the state system.

Definition 3.6. The system (3.64) is said to be completely controllable (c.c.) if for any given value of $x = x_0 = x(z_0)$, and any initial value of $y = y_0 = y(x_0)$, and any final value of $y = y_f$, there exists a finite value $x = x_1 = x(z_1)$, and a control function $u(x)$, $x_0 \leq x \leq x_1$ such that $y(x_1) = y_f$.

According to (2.126), the solution of (3.64) reads

$$\begin{aligned} &y(x(z)) = \\ &\Phi(x(z), x_0)[y_0 + \int_{x_0}^{x(z)} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))u(x(z))d_q x(z)] \end{aligned} \tag{3.65}$$

where $\Phi(a,b) = Y_0(a)Y_0^{-1}(b)$, $Y_0(x(z))$ being the nonsingular matrix solution of the homogeneous system corresponding to the first equation in equations (3.64). Hence, the system is c.c. if for any value $x_0 = x(z_0)$ and any values y_0 and y_f, there exists a finite value x_1 and a q-nonuniform discrete function $u(x)$, $x_0 \le x \le x_1$, such that

$$y_f = y(x_1) =$$
$$\Phi(x_1,x_0)[y_0 + \int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))u(x(z))d_q x(z)] \tag{3.66}$$

It is not difficult to imagine an example of a non c.c. system. The classical one is similar to the following system

$$\mathcal{D}y_1(x(z)) = a_{11}y_1(x(zq^{-\frac{1}{2}})) + a_{12}y_2(x(zq^{-\frac{1}{2}})) + u(x(z))$$
$$\mathcal{D}y_2(x(z)) = a_{22}y_2(x(zq^{-\frac{1}{2}})). \tag{3.67}$$

This system is not c.c. since $y_2(x(z)) = E_{q^{-\frac{1}{2}}}(a_{22}; z_0; z)$; and consequently $u(x(z))$ has no control over it.

The following controllability criterion is valid not only for constant systems but also for varying ones. Moreover, it gives an explicit expression for the control function $u(x)$.

Theorem 3.7. *The system (3.64) is c.c. iff the $k \times k$ symmetric matrix*

$$U(x_0,x_1) =$$
$$\int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))B^T(x(z))\Phi(x_0, x(zq^{\frac{1}{2}}))^T d_q x(z) \tag{3.68}$$

is nonsingular. In the latter case, the control function is given by

$$u(x(z)) = -B^T(x(z))\Phi(x_0, x(zq^{\frac{1}{2}}))^T U^{-1}(x_0, x_1)[y_0 - \Phi(x_0, x_1)y_f]$$
$$x_0 \leq x \leq x_1 \tag{3.69}$$

and transfers $y_0 = y(x_0)$ *to* $y_f = y(x_1)$.

Proof. Necessity. By contradiction: Suppose that the system is c.c. and the matrix $U(x_0, x_1)$ is singular. For an arbitrary k-vector α, we have

$$\begin{aligned}\alpha^T U \alpha &= \int_{x_0}^{x_1} \phi^T(x(z), x_0)\phi(x(z), x_0) d_q x(z) \\ &= \int_{x_0}^{x_1} \| \phi \|^2 \, d_q x(z) \geq 0 \end{aligned} \tag{3.70}$$

where $\phi(x, x_0) = B^T(x)\Phi^T(x_0, x(zq^{\frac{1}{2}}))\alpha$. Thus U is positive semi-definite. It remains to show that the inequality is rigorous. Suppose that there exists $\hat{\alpha} : \hat{\alpha}^T U \hat{\alpha} = 0$. In that case

$$\int_{x_0}^{x_1} \| \hat{\phi} \|^2 \, d_q x(z) = 0 \Rightarrow \hat{\phi} = B^T(x)\Phi^T(x_0, x(zq^{\frac{1}{2}}))\hat{\alpha} = 0 \tag{3.71}$$

$$(\| \hat{\phi} \| = 0 \Leftrightarrow \hat{\phi} = 0). \tag{3.72}$$

As the system is c.c. let $\hat{u}(x)$ be the control that transfers $y(x_0) = \hat{\alpha}$ in $y(x_1) = 0$. We get

$$\hat{\alpha} = -\int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))\hat{u}(x(z)) d_q x(z). \tag{3.73}$$

Hence

$$\| \hat{\alpha} \|^2 = \hat{\alpha}^T \hat{\alpha}$$

$$= -\int_{x_0}^{x_1} \hat{u}^T(x(z)) B^T(x(z)) \Phi^T(x_0, x(zq^{\frac{1}{2}})) \hat{\alpha} d_q x(z) = 0 \quad (3.74)$$

$$\Leftrightarrow \hat{\alpha} = 0. \quad (3.75)$$

Thus U is positive definite hence it is nonsingular.

Sufficiency. If U is nonsingular, the control in (3.69) is defined and we need to show that it transfers $y_0 = y(x_0)$ to $y_f = y(x_1)$. Loading (3.69) in (3.66) gives

$$y(x_1) = \Phi(x, x_0)[y_0$$

$$-(\int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}})) B(x(z)) B^T(x(z)) \Phi^T(x_0, x(zq^{\frac{1}{2}})) d_q x(z)) U^{-1}(x_0, x_1)$$

$$(y_0 - \Phi(x_0, x_1) y_f)]$$

$$= \Phi(x, x_0)[y_0 - (y_0 - \Phi(x_0, x_1) y_f)] = y_f \quad (3.76)$$

and the theorem is proved. □

Note that a scalar control system (k=m=1)

$$\mathcal{D} y(x(z)) = a(x(z)) y(x(zq^{-\frac{1}{2}})) + b(x(z)) u(x(z)) \quad (3.77)$$

is necessarily c.c. since, in such a case, (3.68) is necessarily non singular.

If the system is not c.c., for some y_0 and y_f, there can be or not a control $u(x)$ that joins them. The existence of such a connection control between two given states is given by the following

Theorem 3.8. *If for given* (x_0, y_0) *and* (x_1, y_f)*, there exists a k-vector* γ *such that*

$$U(x_0, x_1)\gamma = y_0 - \Phi(x_0, x_1)y_f \tag{3.78}$$

then the control $u(x) = B^T(x)\Phi^T(x_0, x)\gamma$ *transfers* $y_0 = y(x_0)$ *in* y_f*.*

Proof. Loading $u(x)$ in (3.66) gives

$$\begin{aligned}\Phi(x_1, x_0)[y_0 - (\int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))B^T(x(z))\Phi^T(x_0, x(zq^{\frac{1}{2}}))\gamma] \\ = \Phi(x_1, x_0)[y_0 - (y_0 - \Phi(x_0, x_1)y_f)] = y_f.\end{aligned} \tag{3.79}$$

□

3.2.2 Observability

The concept of observability is closely related to that of controllability. Generally speaking, a system is completely observable if and only if the knowledge of the input and output during a sufficient time suffices to determine the state of the system.

Definition 3.9. The system (3.64) is completely observable (c.o.) if for any x_0, there exists a finite x_1 such that the knowledge of $v(x)$ and $u(x)$ for $x_0 \leq x \leq x_1$ suffices to determine $y_0 = y(x_0)$.

Similarly to Theorem 3.7, the basic observability criterion for time varying systems reads

Theorem 3.10. *The system (3.64) is c.o. iff the $k \times k$ symmetric matrix*

$$V(x_0, x_1) = \int_{x_0}^{x_1} \Phi^T(x(zq^{\frac{1}{2}}), x_0) C^T(x(z)) C(x(z)) \Phi(x(zq^{\frac{1}{2}}), x_0) d_q x(z) \tag{3.80}$$

is nonsingular. In the latter case, we have

$$y_0 = V^{-1}(x_0, x_1) \int_{x_0}^{x_1} \Phi^T(x(zq^{\frac{1}{2}}), x_0) C^T(x) v(x(z)) d_q x(z). \tag{3.81}$$

Proof. Necessity. The proof is similar to the corresponding one in Theorem 3.7.

Sufficiency. Supposing that $u(x) \equiv 0$ (this does not decrease the generalities), $x_0 \leq x \leq x_1$, we have $y(x) = \Phi(x, x_0) y_0$. Hence $v(x) = C(x) y(x(zq^{\frac{1}{2}})) = C(x) \Phi(x(zq^{\frac{1}{2}}), x_0) y_0$. Multiplying on the left by $\Phi^T(x, x_0) C^T(x)$ and integrating from x_0 to x_1, we obtain

$$\begin{aligned} &\int_{x_0}^{x_1} \Phi^T(x(z), x_0) C^T(z) v(x) d_q x(z) \\ &= (\int_{x_0}^{x_1} \Phi^T(x, x_0) C^T(x) C(x) \Phi(x(zq^{\frac{1}{2}}), x_0) d_q x(z)) y_0 \\ &= V(x_0, x_1) y_0. \end{aligned} \tag{3.82}$$

Thus if $V(x_0, x_1)$ is nonsingular, we have

$$y_0 = V^{-1}(x_0, x_1) \int_{x_0}^{x_1} \Phi^T(x(zq^{\frac{1}{2}}), x_0) C^T(x) v(x) d_q x(z). \tag{3.83}$$

□

The controllability and observability are two concepts with physically distinct meanings but that are mathematically equivalent as shows

the following *q-nonuniform duality theorem:*

Theorem 3.11. *The system (3.64) is c.c. iff the dual system*

$$\begin{aligned} \mathcal{D}y(x(z)) &= -A^T(x(z))y(x(zq^{\frac{1}{2}})) + C^T(x(z))u(x(z)) \\ v(x(z)) &= B^T(x(z))y(x(zq^{-\frac{1}{2}})) \end{aligned} \tag{3.84}$$

is c.o. and conversely.

Proof. Considering (3.64),(3.68),(3.80), and (3.84), we remark that to prove the *necessity*, it suffices to prove that if

$$\mathcal{D}\Phi(x(z), x_0) = A(x(z))\Phi(x(zq^{-\frac{1}{2}}), x_0) \tag{3.85}$$

then

$$\mathcal{D}\Phi^T(x_0, x(z)) = -A^T(x(z))\Phi^T(x_0, x(zq^{\frac{1}{2}})). \tag{3.86}$$

From Theorem 2.3, it follows that if $\Phi(x(z), x_0)$ satisfies

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(zq^{-\frac{1}{2}})) \tag{3.87}$$

then its inverse, that is, $\Phi(x_0, x(z))$ satisfies

$$\mathcal{D}Z(x) = -Z(x(zq^{\frac{1}{2}}))A(x(z)).$$

Hence

$$\mathcal{D}\Phi(x_0, x(z)) = -\Phi(x_0, x(zq^{\frac{1}{2}}))A(x(z)). \tag{3.88}$$

Carrying out the transpose on both sides, one gets the required equality (3.86). The *sufficiency* is proved similarly. □

This duality clearly allows to relate results in controllability and observability theories.

3.2.3 Linear optimal controllability

Consider now the q-nonuniform integration functional

$$J(y(x),u(x)) = \int_a^b f^0[x(z),y(x(q^{-\frac{1}{2}}z)),u(x(z))]d_qx(z)$$
$$= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) f^0[x(z),y(x(q^{-\frac{1}{2}}z)),u(x(z))]. \quad (3.89)$$

The functional is defined on the cartesian product $E^k \times U$ where U is the set of admissible (that is which values belong to a fixed set in $\mathbb{R}^k$) functions $u(x)$, and E^k is the set of k-vector functions $y(x)$ satisfying the boundary constraints

$$y(q^{\alpha-\frac{1}{2}}) = C_1; \quad y(q^{\beta+\frac{1}{2}}) = C_2, \quad (3.90)$$

C_1, C_2 - some constants. Consider then the q-nonuniform difference linear control system

$$\mathcal{D}y(x(z)) = A(x(z))y(x(q^{-\frac{1}{2}}z)) + B(x(z))u(x(z)), \quad (3.91)$$

subject to the boundary constraints (3.90). This system may be written in coordinates form as

$$\mathcal{D}y_i(x(z)) = f_i[x(z),y(x(q^{-\frac{1}{2}}z)),u(x(z))],\ i=1,\dots,k. \quad (3.92)$$

where

$$f_i[x(z),y(x(q^{-\frac{1}{2}}z)),u(x(z))] = \sum_{j=1}^{n} a_{ij}(x(z))y_j(x(zq^{-\frac{1}{2}}))$$
$$+ \sum_{j=1}^{m} b_{ij}(x(z))u_j(x(z)),\ i=1,\dots,k. \quad (3.93)$$

The linear optimal control problem consists in finding among all admissible vector functions $u(x)$, that for which the corresponding solution of (3.90), (3.91) (which is supposed to be c.c.) is an extremal of the functional (3.89). The functions $y(x)$ and $u(x)$ are said to constitute a *q-nonuniform optimal process* and are called *q-nonuniform optimal trajectory* and *q-nonuniform optimal control* respectively.

According to the general theory of the first paragraph, the necessary condition for the above linear optimal control problem is given by the *q-nonuniform Hamilton-Pontryaguin system*

$$\mathcal{D}y_i(x) = H_{\psi_i} \tag{3.94}$$

$$\mathcal{D}[\psi_j(x(z))] = -H_{y_j},\ j = 1, \ldots, k \tag{3.95}$$

and

$$H_{u_j} = 0,\ j = 1, \ldots, m. \tag{3.96}$$

where

$$H = f^0[x(z), y(x(q^{-\frac{1}{2}}z)), u(x(z))] + \sum_{i=1}^{n} \psi_i(x(zq^{\frac{1}{2}}))[f_i[x(z), y(x(q^{-\frac{1}{2}}z)), u(x(z))]], \tag{3.97}$$

is the *q-nonuniform Hamilton-Pontryaguin function.*

Using (3.93), (3.97) the system (3.94)-(3.96) reduces to

$$\mathcal{D}y_i(x) = \sum_{j=1}^{n} a_{ij}(x(z))y_j(x(zq^{-\frac{1}{2}})) + \sum_{j=1}^{m} b_{ij}(x(z))u_j(x(z)) \tag{3.98}$$

$$\mathcal{D}[\psi_j(x(z))] = -f^0_{y_j} - \sum_{i=1}^{n} a_{ij}(x(z))\psi_i(x(zq^{\frac{1}{2}})),\ i, j = 1, \ldots, k \tag{3.99}$$

and

$$f_{u_j}^0 + \sum_{i=1}^{m} b_{ij}(x(z))\psi_i(x(zq^{\frac{1}{2}})) = 0,\ j = 1, \ldots, m. \tag{3.100}$$

Example. *The q-nonuniform Ramsey model*:

The continuous Ramsey model [22, 48] consists in finding the maximum of the utility functional

$$\int_0^X e^{-px} v(c(x))dx \tag{3.101}$$

under the constraint

$$w'(x) = rw(x) - c(x) \tag{3.102}$$

where $v(.)$ is the instantaneous utility function, $c(.)$ the consumption, $w(.)$ the production, p the discount rate, r the yield rate.
This problem may be seen as a Lagrange problem or more generally as an optimal control problem. In the latter case, the optimality necessary condition for the problem is the Hamilton-Pontryaguin system

$$\dot{w}(x) = H'_{\psi}, \tag{3.103}$$

$$\dot{\psi}(x) = -H'_{w}, \tag{3.104}$$

$$H'_c = 0. \tag{3.105}$$

where the Hamilton-Pontryaguin function

$$H(x, w(x), \psi(x)) = e^{-px} v(c(x)) + \psi(x)[rw(x) - c(x)], \tag{3.106}$$

is the *total utility* and $\psi(x)$ the *shadow price* (see [22] for the economic sense). (3.103) is nothing but (3.102) while (3.104) and (3.105)

give

$$\frac{d}{dx}v'(c(x)) = (p-r)v'(c(x)). \tag{3.107}$$

We can now generalize the model to the q-nonuniform difference calculus. Since (3.102) can be written as

$$c(x) = -e^{rx}[e^{-rx}w(x)]', \tag{3.108}$$

the q-nonuniform difference Ramsey model consists in finding the maximum of the utility functional

$$\int_0^X E_{q^{-\frac{1}{2}}}(-p; z_0; z)v(c(x(z)))dx(z) \tag{3.109}$$

under the constraint

$$c(x(z)) = -E_{q^{-\frac{1}{2}}}(r; z_0; q^{\frac{1}{2}}z)\mathcal{D}\left[E_{q^{\frac{1}{2}}}(-r; z_0; z)w(x(z))\right]. \tag{3.110}$$

By the rule of q-nonuniform derivation of a product and the formula

$$\mathcal{D}E_{q^{\frac{1}{2}}}(a; z_0; z) = aE_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z), \tag{3.111}$$

the constraint (3.110) simplifies in the q-nonuniform difference linear control system

$$\mathcal{D}w(x(z)) = rw(x(zq^{-\frac{1}{2}})) - c(x(z)), \tag{3.112}$$

which is c.c. since it is a scalar one, as (3.77). Then, the q-nonuniform Hamilton-Pontryaguin system reads

$$\mathcal{D}w(x(z)) = H'_{\psi}, \tag{3.113}$$

$$\mathcal{D}\psi(x(z)) = -H'_{w}, \tag{3.114}$$

$$H'_{c} = 0, \tag{3.115}$$

where

$$H(x,w(x),\psi(x)) = E_{q^{q^{-\frac{1}{2}}}}(-p;z_0;z)v(c(x(z))) \\ +\psi(x(q^{\frac{1}{2}}z))[rw(x(q^{-\frac{1}{2}}z)) - c(x(z))], \tag{3.116}$$

is the the q-nonuniform Hamilton-Pontryaguin function.
(3.113) is nothing but (3.112) while (3.114) and (3.115) give respectively the equations

$$\mathcal{D}\psi(x(z)) = -r\psi(x(q^{\frac{1}{2}}z)), \tag{3.117}$$

$$\psi(x(q^{\frac{1}{2}}z)) = E_{q^{-\frac{1}{2}}}(-p;z_0;z)v'(c(x(z))). \tag{3.118}$$

Combining (3.117), (3.118) and using the formula

$$\mathcal{D}E_{q^{-\frac{1}{2}}}(-a;z_0;z) = -aE_{q^{-\frac{1}{2}}}(-a;z_0;q^{-\frac{1}{2}}z), \tag{3.119}$$

we obtain

$$\mathcal{D}\hat{v}(z) = p\hat{v}(zq^{-\frac{1}{2}}) - r\hat{v}(zq^{\frac{1}{2}});\ \hat{v}(z) = v'(c(x(zq^{-\frac{1}{2}}))), \tag{3.120}$$

a *q-nonuniform version of the optimality condition* (3.107). This equation clearly generalizes the equation (40) in [48] since the Askey-Wilson derivative generalizes the Hahn quantum derivative (see e.g. [43, 44, 50]).

Chapter 4

q-Nonuniform difference multiple orthogonal polynomials

4.1 The Askey-Wilson polynomials

We consider the second-order divided difference operator [50]:

$$\sigma(s)\frac{\Delta}{\Delta t(s-\frac{1}{2})}.\frac{\nabla}{\nabla t(s)}+\tau(s)\frac{\Delta}{\Delta t(s)} \tag{4.1}$$

where

$$\begin{aligned}\sigma(s)&=\hat{\sigma}(t(s))-\tfrac{1}{2}\hat{\tau}(t(s))\Delta t(s-\tfrac{1}{2})\\ \tau(s)&=\hat{\tau}(t(s));\end{aligned} \tag{4.2}$$

$\hat{\sigma}$ and $\hat{\tau}$ being polynomials of degree ≤ 2 and 1 respectively,

$$t(s)=c_1q^s+c_2q^{-s}+c_3 \tag{4.3}$$

the lattice, and $\Delta h(s) = h(s+1) - h(s); \nabla h(s) = h(s) - h(s-1)$. The form of the operator in (4.1) being invariant under the transformations

$$t \longrightarrow At + B; \qquad s \longrightarrow s - s_0, \tag{4.4}$$

we can use them to transform the lattice in (4.3) in its canonical (see (8)) form

$$t(s) = \frac{q^s + q^{-s}}{2}. \tag{4.5}$$

In that case, up to a multiplication by a constant, the operator in (4.1) can, after simple computations, be reduced to [11]:

$$\pounds = \frac{1}{z - z^{-1}} \left(\mathcal{A}(z)\mathbf{E}_q - [\mathcal{A}(z) + \mathcal{B}(z)] + \mathcal{B}(z)\mathbf{E}_q^{-1} \right) \tag{4.6}$$

where

$$\begin{gathered}
\mathcal{A}(z) = \frac{A_{-2}z^{-2} + A_{-1}z^{-1} + A_0 + A_1 z + A_2 z^2}{qz - z^{-1}} \\
\mathcal{B}(z) = \frac{A_2 z^{-2} + A_1 z^{-1} + A_0 + A_{-1} z + A_{-2} z^2}{z - qz^{-1}} \\
A_{-2} = 1; A_{-1} = -(a+b+c+d); A_0 = ab + ac + ad + bc + bd + cd \\
A_1 = -(abc + abd + bcd + acd); A_2 = abcd \\
z = q^s \\
a = q^{s_1}; b = q^{s_2}; c = q^{s_3}; d = q^{s_4} \\
\mathbf{E}_q^i(k(z)) = k(q^i z), i \in Z,
\end{gathered} \tag{4.7}$$

s_1, s_2, s_3 and s_4 being the (mutually-different) roots of $\sigma(s)$.
The operator $\pounds$ can also be written as [38]:

$$\begin{gathered}
\pounds(z) = \upsilon(z)\mathbf{E}_q - (\upsilon(z) + \upsilon(z^{-1})) + \upsilon(z^{-1})\mathbf{E}_q^{-1}, \\
\upsilon(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.
\end{gathered} \tag{4.8}$$

Letting $\mathcal{D}$, be the Askey-Wilson first-order divided difference operator (11), one can also write $\pounds$ as:

$$\pounds = \left[\tfrac{(q-1)^2}{4q}\tfrac{z^2-1}{z\omega(z)}\mathcal{D}\tfrac{z^2-1}{z}\upsilon(q^{-\frac{1}{2}}z)\omega(q^{-\frac{1}{2}}z)\right]\mathcal{D}, \tag{4.9}$$

where

$$\tfrac{\omega(qz)}{\omega(z)} = \tfrac{\upsilon(z)}{\upsilon((qz)^{-1})}. \tag{4.10}$$

The Askey-Wilson polynomials (AWP) [9], $\mathcal{P}_n(x(z))$, are defined by:

$$\mathcal{P}_n(x(z)) = \frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}\right| q;q\right) \tag{4.11}$$

where the basic hypergeometric (or q-hypergeometric) series ${}_r\phi_s$ read:

$${}_r\phi_s\left(\left.\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\right| q;z\right) := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k}$$

with

$$(a_1,\ldots,a_r;q)_k := (a_1;q)_k\ldots(a_r;q)_k,$$

while

$$(\sigma;q)_0 := 1, (\sigma;q)_k := \prod_{i=0}^{k-1}(1-\sigma q^i), k = 1,2,3,\ldots \tag{4.12}$$

and satisfy the second order q-(nonuniform) difference equation:

$$\pounds\mathcal{P}_n(x(z)) = \lambda(n)\mathcal{P}_n(x(z)) \tag{4.13}$$

where

$$\lambda(n) = -(1 - q^{-n})(1 - abcdq^{n-1}). \tag{4.14}$$

The operator $\pounds$ was shown in [35] to be self-adjoint (more precisely, the operator in square brackets, in the r.h.s. of (4.9), is adjoint to the first-order Askey-Wilson divided difference operator) in the space $S_{a,b,c,d}$ of real polynomials in $x(z)$ with the inner product

$$(h_1, h_2) := \frac{1}{2\pi i} \oint_C h_1(x(z))h_2(x(z))\omega(z)\frac{dz}{z}, \tag{4.15}$$

where $\omega(z)$ is given by (4.10) and C is a deformation of the unit circle.

Considering the representation in (4.9) and the equation in (4.13), we will refer to $\pounds$, everywhere in this work, as the *Askey-Wilson second order q-(nonuniform) difference operator.*

In this section, we apply the factorization method discussed in [11], to the operator $\pounds$.

The factorization that we are going to carry out for the Askey-Wilson second order q-(nonuniform) difference operator will furnish various formulas such as *strict* (i.e. without any perturbation of parameters) difference relations, recurrence relations and Rodrigues type formulas for the general sequences of eigenfunctions of the Askey-Wilson second order q-(nonuniform) difference operator, including the sequence of the Askey-Wilson polynomials. Such formulas appeared firstly in [11].

As already noted, the Askey-Wilson polynomials are polynomial solutions the following second order q-(nonuniform) difference equation [38]:

$$\pounds \mathcal{P}_n(x(z)) = \lambda(n)\mathcal{P}_n(x(z)). \tag{4.16}$$

In anticipation on the coming q-factorization, we are led to reformulate, before the factorization, the equation (4.16) as follows:

$$\{\mathcal{A}(qz)\mathbf{E}_q^2 - [\mathcal{A}(qz) + \mathcal{B}(qz) + \mathcal{K}(qz)\lambda(n)]\mathbf{E}_q + \mathcal{B}(qz)\}\mathcal{P}_n(x(z)) = 0 \tag{4.17}$$

where $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are given in (4.7) and $\mathcal{K}(z) = z - z^{-1}$.

Consider next the following q-difference operator

$$\mathcal{H}(z;n) = \mathcal{A}(z)\left[\mathcal{A}(qz)\mathbf{E}_q^2 - [\mathcal{A}(qz) + \mathcal{B}(qz) + \mathcal{K}(qz)\lambda(n)]\mathbf{E}_q + \mathcal{B}(qz)\right]. \tag{4.18}$$

The central result of this section is the following

Theorem 4.1. *The operator $\mathcal{H}$ in (4.18) factorizes into*

$$\begin{aligned}\mathcal{H}(z;n) - \mu(n) &= (\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}(z;n))(\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}(z;n)) \\ \mathcal{H}(z;n+1) - \mu(n) &= (\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}(z;n))(\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}(z;n))\end{aligned} \tag{4.19}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mu(n)$ are such that

$$\mu(-1) = 0 \tag{4.20}$$

and

$$\mathcal{F}(z;n) - \mathcal{G}(z;n-1) = C_1(n)x(z) + C_2(n). \tag{4.21}$$

Proof. As one can verify, the operatorial relations (4.19) are equivalent to the system:

$$\begin{aligned} \mathcal{F}(qz;n)+\mathcal{G}(z;n) &= -(\mathcal{A}(qz)+\mathcal{B}(qz)+\mathcal{K}(qz)\lambda(n)) \\ \mathcal{F}(z;n)\mathcal{G}(z;n) &= \mathcal{A}(z)\mathcal{B}(qz)-\mu(n) \\ \Delta_q(\mathcal{F}(z;n)-\mathcal{G}(z;n)) &= (\lambda(n+1)-\lambda(n))\mathcal{K}(qz). \end{aligned} \tag{4.22}$$

Using the q-integration, we first transform the system (4.22) in :

$$\begin{aligned} \mathcal{F}(z;n)+\mathcal{F}(qz;n) = \; & -(\mathcal{A}(qz)+\mathcal{B}(qz))-(\beta_{-1}-\tfrac{\lambda(n)}{q})z^{-1}-\beta_0 \\ & -(\beta_1+\lambda(n)q)z \\ \mathcal{F}(z;n)\mathcal{G}(z;n) = \; & \mathcal{A}(z)\mathcal{B}(qz)-\mu(n) \\ \mathcal{G}(z;n)-\mathcal{F}(z;n) = \; & \sum_{i=-1}^{i=1}\beta_i z^i \end{aligned} \tag{4.23}$$

where $\beta_{-1}=\frac{\lambda(n+1)-\lambda(n)}{1-q}$; $\beta_1=q\beta_{-1}$; β_0 remaining arbitrary for the moment.

Observing the first equation in (4.23) and then using the last one, it becomes sensible to search $\mathcal{F}(z;n)$ and then $\mathcal{G}(z;n)$ under the forms:

$$\begin{aligned} \mathcal{F}(z;n) := \; & \tfrac{F_{-2}z^{-2}+F_{-1}z^{-1}+F_0+F_1z+F_2z^2}{qz-z^{-1}} \\ \mathcal{G}(z;n) := \; & \tfrac{(F_{-2}-\beta_{-1})z^{-2}+(F_{-1}-\beta_0)z^{-1}+F_0+(F_1+\beta_0 q)z+(F_2+\beta_1 q)z^2}{qz-z^{-1}}. \end{aligned} \tag{4.24}$$

Taking μ, β_0, F_{-2}, F_{-1}, F_0, F_1, F_2 as unknowns, the system (4.22) will then be transformed in an algebraic system of 16 equations for 7 unknowns. To solve it, one first determines those unknowns from 7 equations and then ensures himself that they satisfy all the remaining 9 equations.

The result is:

$$
\begin{aligned}
F_{-2}(n): &\quad \tfrac{\lambda(n)-q\lambda(n+1)}{q^2-1} - \tfrac{q+A_2}{q^2+q} \\
F_2(n): &\quad \tfrac{\lambda(n)q-\lambda(n+1)}{1-q^2}q^2 - \tfrac{q^2+qA_2}{q+1} \\
\beta_0(n): &\quad \tfrac{1-q}{(\lambda(n)-\lambda(n+1))q^3}\{(2\tfrac{\lambda(n)q-\lambda(n+1)}{1-q^2}q^2 + \tfrac{\lambda(n+1)-\lambda(n)}{1-q}q^2 \\
&\quad -2\tfrac{q^2+qA_2}{1+q})(A_1+qA_{-1}) + (2A_1q^2 + 2A_2A_{-1}q)\} \\
F_{-1}(n): &\quad \tfrac{\beta_0(n)}{2} - \tfrac{A_1+qA_{-1}}{2q} \\
F_1(n): &\quad -\tfrac{q\beta_0(n)}{2} - \tfrac{A_1+qA_{-1}}{2} \\
F_0(n): &\quad \tfrac{1}{q+q^2}\{q^2 - q^3 - A_0(q+q^2) + A_2(q-1) + q^2(\lambda(n)+\lambda(n+1))\} \\
\mu(n): &\quad A_0 + A_1A_{-1}q^{-1} + A_0A_2q^{-2} + F_0(n)\beta_{-1}(n) + F_{-1}(n)\beta_0(n) \\
&\quad -2F_{-2}(n)F_0(n) - F_{-1}^2(n).
\end{aligned}
\tag{4.25}
$$

Thus, (4.24) is determined and a solution of the factorization system (4.22) is found. It remains to prove (4.20) and (4.21). Direct computations allow to confirm the equation (4.20) and to get the following identity

$$
\begin{aligned}
&\mathcal{F}(z;n) - \mathcal{G}(z;n-1) = \\
&[2A_2q^{-1}q^n - 2q^{-n}]x(z) - \tfrac{\beta_0(n)+\beta_0(n-1)}{2}, \quad (\forall n \in Z).
\end{aligned}
\tag{4.26}
$$

that is the relation (4.21) with

$$
\begin{aligned}
C_1(n) &= 2A_2q^{-1}q^n - 2q^{-n} \\
C_2(n) &= -\tfrac{\beta_0(n)+\beta_0(n-1)}{2}.
\end{aligned}
\tag{4.27}
$$

□

A direct verification allows next to establish the following useful relations

$$\begin{aligned} \mathcal{F}(z;-1) &= -\mathcal{B}(qz) \\ \mathcal{G}(z;-1) &= -\mathcal{A}(z), \end{aligned} \tag{4.28}$$

$$(\mathcal{A}(z)\mathbf{E}_q + \mathcal{G}(z;-1))(1) = 0 \tag{4.29}$$

and

$$\begin{aligned} (A(z)\mathbf{E}_q + \mathcal{F}(z;0))(1) = A(z) + \mathcal{F}(z;0) = \mathcal{F}(z,0) - \mathcal{G}(z,-1) \\ = [2A_2q^{-1} - 2]x(z) - \tfrac{\beta_0(0)}{2} + \tfrac{A_1 - qA_{-1}}{2q}. \end{aligned} \tag{4.30}$$

Note also that since $\lambda(0) = 0$ the polynomial $\mathcal{P}_0(x(z)) = 1$ is a solution of the equation (4.16) for n=0, or equivalently the equation $\mathcal{H}(z,0)\mathcal{P}_0(x(z)) = 0$. Now, considering the preceding, we obtain the corollary of the theorem 4.1:

Corollary 4.2. *The AWP $\mathcal{P}_n(x(z))$, polynomial solutions of (4.16) satisfy the difference relations*

$$\mathcal{P}_{n+1}(x(z)) = (A(z)\mathbf{E}_q + \mathcal{F}(z;n))\mathcal{P}_n(x(z)) \tag{4.31}$$

$$-\mu(n-1)\mathcal{P}_{n-1}(x(z)) = (A(z)\mathbf{E}_q + \mathcal{G}(z;n-1))\mathcal{P}_n(x(z)) \tag{4.32}$$

and the three-term recurrence relations

$$\begin{gathered} \mathcal{P}_{n+1}(x(z)) + \mu(n-1)\mathcal{P}_{n-1}(x(z)) = \\ \left([2A_2q^{-1}q^n - 2q^{-n}]x(z) - \tfrac{\beta_0(n)+\beta_0(n-1)}{2}\right)\mathcal{P}_n(x(z)) \\ \mathcal{P}_0(x(z)) = 1; \\ \mathcal{P}_1(x(z)) = [2A_2q^{-1} - 2]x(z) - \tfrac{\beta_0(0)}{2} + \tfrac{A_1 - qA_{-1}}{2q}. \end{gathered} \tag{4.33}$$

The monic AWP polynomials $\mathcal{P}_n^\circ(x(z)) = \mathcal{P}_n(x(z))/\varrho(n)$ where

$$\varrho(n+1)/\varrho(n) = C_1(n) \tag{4.34}$$

satisfy

$$\mathcal{P}_{n+1}^\circ + a_n^2 \mathcal{P}_{n-1}^\circ = (x(z) - b_n)\mathcal{P}_n^\circ \tag{4.35}$$

where

$$a_n^2 = \frac{\mu(n-1)}{C_1(n)C_1(n-1)}; \qquad b_n = -C_2(n)/C_1(n). \tag{4.36}$$

From the difference relations (4.31), one deduces especially a Rodrigues type formula:

$$\tilde{c}(n)\mathcal{P}_n(\chi(z)) = \prod_{i=0}^{n-1}[\mathcal{A}(z)\mathbf{E}_q + \mathcal{F}(z;i)](1), n = 1, 2, 3, \ldots \tag{4.37}$$

or equivalently

$$\tilde{c}(n)\mathcal{P}_n(\chi(z)) = \frac{1}{\rho(z)}\prod_{i=0}^{n-1}[\mathbf{E}_q + \mathcal{F}(z;i)](\rho(z)), n = 1, 2, 3, \ldots \tag{4.38}$$

where

$$\frac{\rho(qz)}{\rho(z)} = \mathcal{A}(z). \tag{4.39}$$

Note that the relations (4.22), (4.26), (4.27) and (4.36) lead to the following relations for the AWP:

$$\begin{gathered}
\mathcal{F}(z,n) = \mathcal{G}(z,n-1) + (x(z) - b_n)C_1(n) \\
\mathcal{G}(z,n) = \mathcal{F}(z,n-1) - (x(qz) - b_n)C_1(n)) \\
\mathcal{F}(z,n)\mathcal{G}(z,n) = \mathcal{F}(z,n-1)\mathcal{G}(z,n-1) \\
+a_n^2 c_1(n-1)C_1(n) - a_{n+1}^2 C_1(n)C_1(n+1).
\end{gathered} \tag{4.40}$$

This means that the AWP are a special case of Laguerre-hahn polynomials [43] being polynomials 0-associated (due to that $\mu(-1) = 0$) to classical polynomials.

On the other side by the consideration above, we have the following

Theorem 4.3. *The AWP $\mathcal{P}_n(x(z))$, $n = 0, 1, \ldots$ satisfy the second order q-difference equation*

$$\{\mathcal{G}(qz;-1)\mathbf{E}_q^2 - [\mathcal{F}(qz;n) + \mathcal{G}(z;n)]\mathbf{E}_q + \mathcal{F}(z;-1)\}\mathcal{P}_n(x(z)) = 0, \qquad (4.41)$$

where $\mathcal{F}$ and $\mathcal{G}$ satisfy the relations (4.40).

This result confirms the fact that the AWP are a special case of the semi-classical polynomials [44] on lattices (8). Indeed, the semi-classical polynomials are known to satisfy a type (4.41) difference equation, with $\mathcal{F}$ and $\mathcal{G}$ satisfying relations generalizing (4.40).

Functions generalizing the Askey-Wilson polynomials

We envisage now to show how to produce a sequence of functions $\Phi_n(z,t)$ generalizing the AWP in that sense that they are eigenfunctions of the Askey-Wilson second order q-(nonuniform) difference operator $\pounds$ and

$$\Phi_n(z,t) \longrightarrow \mathcal{P}_n(\chi(z)), \qquad t \longrightarrow 1. \qquad (4.42)$$

We showed above that if the eigenvalue of the Askey-Wilson second order q-(nonuniform) difference operator is $\lambda(n) = -(1-q^{-n})(1-abcdq^{n-1})$, then a factorization as in (4.19) is realizable. Now, using (4.22), one finds that conversely, the possibility of a type (4.19) factorization implies necessarily that the eigenvalue in action is ${}^t\lambda(n) =$

$-(1-tq^{-n})(1-abcdt^{-1}q^{n-1})$. On the other side, it is clear that the presence of the parameter t in ${}^t\lambda(n)$ in no way annoys the factorization (4.19) (thanks to the specific dependence of ${}^t\lambda(n)$ at t). This means that if in the expressions (4.25), we substitute $\lambda(n)$ by ${}^t\lambda(n)$, the resulting expressions for $\mathcal{F}$ and $\mathcal{G}$, say ${}^t\mathcal{F}$ and ${}^t\mathcal{G}$, solve a type (4.19) factorization problem for an operator, say ${}^t\mathcal{H}(z;n)$, obtained from $\mathcal{H}(z;n)$ replacing $\lambda(n)$ by ${}^t\lambda(n)$.
Let us consider next the equation:

$$\pounds y(z,t) =^t \lambda(0)y(z,t), \tag{4.43}$$

or equivalently

$${}^t\mathcal{H}(z;0)y(z,t)=0. \tag{4.44}$$

This equation is not new: A little wider equation (where the Askey-Wilson second order q-(nonuniform) difference operator is replaced by the corresponding operator in (4.1)) has been explicitly solved in [10]. In our situation, according to [10], one first solves the non-homogeneous equations:

$$\left(\pounds -^t \lambda(0)\right) u(z,t,\alpha) = G(z,t,\alpha) \tag{4.45}$$

where

$$G(z,t,\alpha) = \frac{(\alpha t, abcd\alpha t^{-1}q^{-1};q)_\infty}{\alpha(ab\alpha, ac\alpha, ad\alpha, \alpha q;q)_\infty}(az, az^{-1};q)_\infty \tag{4.46}$$

and $(\sigma;q)_\infty := \lim_{i\to\infty}(\sigma;q)_i$, for the following values of α:

$$\alpha := 1 \qquad \alpha := \frac{q}{ab} \qquad \alpha := \frac{q}{ac} \qquad \alpha := \frac{q}{ad}.$$

The corresponding solutions are the functions [10]:

$$u(z,t,\alpha)=\frac{(az,az^{-1};q)_\infty}{(a\alpha z,a\alpha z^{-1};q)_\infty}\sum_{i=0}^{\infty}\frac{(\alpha t,abcd\alpha t^{-1}q^{-1},a\alpha z,a\alpha z^{-1};q)_i}{(ab\alpha,ac\alpha,ad\alpha,\alpha q;q)_i}q^i. \quad (4.47)$$

Let us remark that the AWP appear here as the functions $u(z,q^{-n},1)$, $n=0,1,2,\ldots$

The solutions of the homogeneous equation (4.43) or (4.44) are then obtained by opering adequate (considering (4.46)) linear combinations of any two of the functions (4.47).

Among the solutions of the homogenous equation (4.43) or (4.44), we are interested in those having a constant limit when t converges to the unit. Let us take the "adequate" linear combination of $u(z,t,1)$ and $u(z,t,\alpha)$, $\alpha\neq 1$:

$$y(z,t):=u(z,t,1)-\alpha\frac{(t,abcdt^{-1}q^{-1},ab\alpha,ac\alpha,ad\alpha,\alpha q;q)_\infty}{(ab,ac,ad,q,\alpha t,abcd\alpha t^{-1}q^{-1};q)_\infty}u(z,t,\alpha). \quad (4.48)$$

It is easily seen that $y(z,1)=1$. However, if $\tilde{y}(z,t)$ is another "adequate" linear combination of $u(z,t,\alpha_1)$ and $u(z,t,\alpha_2)$ where α_1 and α_2 differ both from the unit and each other, then $\tilde{y}(z,1)\neq$ *constant*. In other words $\tilde{y}(z,1)$ is the nonconstant solution of the equation (4.16) for $n=0$.

Our point here resides in the following

Theorem 4.4. *If $y(z,t)$ is the solution in (4.48), of (4.43) or (4.44), then the functions*

$$\begin{aligned}\Phi_0(z,t) &:= y(z,t),\\ \Phi_n(z,t) &:= \prod_{i=0}^{n-1}[\mathcal{A}(z)\mathbf{E}_q +^t \mathcal{F}(z;i)]y(z,t),\\ &\quad n=1,2,3,\ldots,\end{aligned} \quad (4.49)$$

generalize the AWP in that sense that they are eigenfunctions of the Askey-Wilson second order q-(nonuniform) difference operator £

(corresponding to ${}^t\lambda(n)$) and they converge to them when t converges to the unit.

Proof. As already noted, the operator ${}^t\mathcal{H}$ admits a type (4.19) factorization but now with $\mathcal{F}$ and $\mathcal{G}$ replaced by ${}^t\mathcal{F}$ and ${}^t\mathcal{G}$. Consequently, the fact that the functions in (4.49) are eigenfunctions of the Askey-Wilson second order q-difference operator $\pounds$ corresponding to ${}^t\lambda(n)$, is a consequence of that $\Phi_0(z;t)$ is an eigenvector of $\pounds$ corresponding to ${}^t\lambda(0)$. To be assured that the functions in (4.49) converge to the AWP (of course, up to a multiplication by a constant), we need essentially to remember that ${}^t\lambda(n)$ converges to $\lambda(n)$ when t converges to the unit and then compare the r.h.s. of (4.49) and (4.37), which proves the theorem. □

4.2 Multiple Askey-Wilson polynomials and related basic hypergeometric multiple orthogonal polynomials

4.2.1 Introduction

In this paragraph (see also [52]) we will first show in Section 4.2.2 how some families of basic hypergeometric polynomials are related by a linear transformation. This transformation is a q-analogue of the Fourier–Jacobi transform that maps Jacobi polynomials to Wilson polynomials [40]. We will consider three families of basic hypergeometric polynomials: the Al-Salam–Chihara polynomials, the continuous dual q-Hahn polynomials and the Askey–Wilson polynomials, and show how they can be obtained by a linear transformation from the little q-Laguerre and the little q-Jacobi polynomials. We then

extend this procedure to multiple orthogonal polynomials in Section 4.2.3. We will first recall multiple little q-Laguerre polynomials in Section 4.2.3.1 and multiple little q-Jacobi polynomials in Section 4.2.3.2 and then apply the linear transformations to obtain multiple Al-Salam–Chihara polynomials (Section 4.2.3.3), multiple continuous dual q-Hahn polynomials (Section 4.2.3.4) and finally multiple Askey-Wilson polynomials (Section 4.2.3.5).

4.2.1.1 Basic hypergeometric orthogonal polynomials

Al-Salam–Chihara polynomials $p_n(x;a,b|q)$ satisfy the orthogonality

$$\int_{-1}^{1} p_n(x;a,b|q)p_m(x;a,b|q)w(\theta;a,b|q)\frac{dx}{\sqrt{1-x^2}} = 0, \qquad n \neq m,$$

where $x = \cos\theta$ and

$$w(\theta;a,b|q) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta};q)_\infty}, \tag{4.50}$$

with parameters a, b satisfying $|a|, |b| < 1$, see [37, §14.8], [31, §15.1]. They are given by

$$p_n(x;a,b|q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}\right| q, q\right). \tag{4.51}$$

Continuous dual q-Hahn polynomials $p_n(x;a,b,c|q)$ satisfy the orthogonality relations

$$\int_{-1}^{1} p_n(x;a,b,c|q)p_m(x;a,b,c|q)w(\theta;a,b,c|q)\frac{dx}{\sqrt{1-x^2}} = 0, \qquad n \neq m,$$

where $x = \cos\theta$ and

$$w(\theta; a, b, c|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}; q)_\infty}, \tag{4.52}$$

with parameters a, b, c satisfying $|a|, |b|, |c| < 1$. They have the basic hypergeometric expression

$$p_n(x; a, b, c|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix} \middle| q, q \right). \tag{4.53}$$

Askey–Wilson polynomials $p_n(x; a, b, c, d|q)$ satisfy the orthogonality relations

$$\int_{-1}^{1} p_n(x; a, b, c, d|q) p_m(x; a, b, c, d|q) w(\theta; a, b, c, d|q) \frac{dx}{\sqrt{1-x^2}} = 0,$$

$$n \neq m,$$

where $x = \cos\theta$ and

$$w(\theta; a, b, c, d|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}, \tag{4.54}$$

with parameters a, b, c, d satisfying $|a|, |b|, |c|, |d| < 1$, [37, §14.1], [31, §15.2]. They are given by

$$p_n(x; a, b, c, d|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q, q \right). \tag{4.55}$$

These three families are connected and are in fact all Askey–Wilson polynomials, since

- $p_n(x; a, b, c, 0|q) = p_n(x; a, b, c|q)$,
- $p_n(x; a, b, 0, 0|q) = p_n(x; a, b|q) = p_n(x; a, b, 0|q)$.

- $a^n H_n(x;a|q) = p_n(x;a,0,0,0|q)$.

The latter polynomials $H_n(x;a|q)$ are the continuous big q-Hermite polynomials which we will use in Section 4.2.2.3. We have normalized the polynomials so that $p_n(0;a,b,c,d|q) = p_n(0;a,b,c|q) = p_n(0;a,b|q) = 1$. Observe that

$$A_k(x) = (ae^{i\theta}, ae^{-i\theta};q)_k = \prod_{j=0}^{k-1}(1 + a^2q^{2j} - 2aq^j x) \tag{4.56}$$

is a polynomial of degree k, so the polynomials (4.51), (4.53), and (4.55) are expressed as a linear combination of these polynomials $A_k(x)$.

Two other families of basic hypergeometric orthogonal polynomials that we will encounter are orthogonal with respect to a discrete measure supported on the q-lattice $\{q^k, k = 0,1,2,\ldots\}$. They are the little q-Laguerre polynomials $q_n(x;a|q)$ [37, §14.20] for which $0 < aq < 1$ and

$$\sum_{k=0}^{\infty} q_n(q^k;a|q)q_m(q^k;a|q)\frac{(aq)^k}{(q;q)_k} = 0, \qquad m \neq n, \tag{4.57}$$

and the little q-Jacobi polynomials $q_n(x;a,b|q)$ [37, §14.12], for which $0 < aq < 1$, $bq < 1$ and

$$\sum_{k=0}^{\infty} q_n(q^k;a,b|q)q_m(q^k;a,b|q)\frac{(aq)^k(bq;q)_k}{(q;q)_k} = 0, \qquad m \neq n. \tag{4.58}$$

They are given by

$$q_n(x;a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}\middle| q, qx\right) = \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(aq;q)_k(q;q)_k}(qx)^k, \tag{4.59}$$

and

$$q_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\middle|\, q, qx\right) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (abq^{n+1};q)_k}{(aq;q)_k (q;q)_k}(qx)^k. \tag{4.60}$$

Here we used the normalization $q_n(0;a|q) = 1 = q_n(0;a,b|q)$.

4.2.1.2 Multiple orthogonal polynomials

Multiple orthogonal polynomials are polynomials in one variable that have orthogonality conditions with respect to several measures. There are two types of multiple orthogonal polynomials, but in this paper we only consider type II multiple orthogonal polynomials. Let $r \geq 1$ be a positive integer and $(\mu_1, \ldots, \mu_r)$ positive measures on the real line for which all the moments exist. We will use multi-indices $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and denote their size by $|\vec{n}| = n_1 + n_2 + \cdots + n_r$. Type II multiple orthogonal polynomials $P_{\vec{n}}(x)$ for the multi-index $\vec{n}$ are monic polynomials of degree $|\vec{n}|$ that satisfy the orthogonality conditions

$$\int P_{\vec{n}}(x) x^k \, d\mu_j(x) = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$. This gives a system of $|\vec{n}|$ homogeneous equations for the $|\vec{n}|$ unknown coefficients of $P_{\vec{n}}$. If the solution exists and if it is unique, then we say that $\vec{n}$ is a normal index. See [31, Ch. 23], [51, §4.3], [1], [45] for a background on multiple orthogonal polynomials.

During the past few decades, various examples of multiple orthogonal polynomials with classical weights have been worked out. Often one can take the orthogonality measures for classical orthogonal polynomials and by allowing r different parameters one gets r measures with

respect to which one can look for the corresponding multiple orthogonal polynomials, see, e.g., [2, 7, 58]. Some of these 'classical' multiple orthogonal polynomials play an important role in applications, e.g., multiple Hermite polynomials and multiple Laguerre polynomials are used in the analysis of random matrices [24, 25, 42] or special determinantal processes [41], multiple Jacobi polynomials and multiple little q-Jacobi polynomials are used in irrationality proofs [54–56], multiple Charlier and multiple Meixner polynomials are used to describe non-Hermitian oscillator Hamiltonians [46, 47, 49], and in general multiple orthogonal polynomials they are useful in the analysis of multidimensional Schrödinger equations and the multidimensional Toda lattice [3, 4].

Beckermann et al. [23] worked out the most general family of classical multiple orthogonal polynomials by giving the multiple Wilson polynomials. These Wilson polynomials are on top of the Askey table [37, p. 183] and from this family one can move to other families of classical multiple orthogonal polynomials by taking limits. They used a transformation (the Fourier-Jacobi transform) that maps Jacobi polynomials to Wilson polynomials (Koornwinder [40]) and showed that this transform allows to generate multiple Wilson polynomials from certain multiple Jacobi polynomials (the Jacobi-Piñeiro polynomials). In this paper we will look at the q-analogue of the Askey table [37, p. 413]. Some multiple q-orthogonal polynomials have already been obtained, such as the multiple little q-Jacobi polynomials [53], multiple q-Charlier polynomials [8] and multiple q-Hahn polynomials [6]. On top of the q-analogue of the Askey table are the Askey–Wilson polynomials and the q-Racah polynomials. In this paper we will obtain multiple Askey–Wilson polynomials (Section 4.2.3.5)

by use of a linear transformation that maps little q-Jacobi polynomials to Askey–Wilson polynomials. We will also obtain multiple continuous dual q-Hahn polynomials (Section 4.2.3.4) and multiple Al-Salam–Chihara polynomials (Section 4.2.3.3) using the transform that maps little q-Laguerre polynomials to continuous dual q-Hahn polynomials and Al-Salam–Chihara polynomials. to achieve this, we first work out the multiple little q-Laguerre polynomials in Section 4.2.3.1 and the multiple little q-Jacobi polynomials in Section 4.2.3.2.

4.2.2 A mapping between basic hypergeometric polynomials

The Al-Salam–Chihara polynomials and the Askey–Wilson polynomials are most naturally expressed in the basis $\{A_k(x); k = 0, 1, 2, \ldots\}$ of polynomials, given in (4.56). Let us also consider the polynomials

$$B_k(x) = (be^{i\theta}, be^{-i\theta}; q)_k = \prod_{j=0}^{k-1}(1 + b^2q^{2j} - 2bq^j x),$$

then the orthogonality of the Al-Salam–Chihara polynomials is equivalent to

$$\frac{1}{2\pi}\int_0^\pi p_n(x; a, b|q)(be^{i\theta}, be^{-i\theta}; q)_j w(\theta; a, b|q)\, d\theta = 0, \qquad 0 \le j \le n-1,$$

and the orthogonality of the Askey–Wilson polynomials is

$$\frac{1}{2\pi}\int_0^\pi p_n(x; a, b, c, d|q)(be^{i\theta}, be^{-i\theta}; q)_j w(\theta; a, b, c, d|q)\, d\theta = 0,$$

$$0 \le j \le n - 1.$$

We can express these polynomials, up to a multiplicative factor, as a determinant:

$$p_n(x) = C_n \det \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,0} & m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-1,0} & m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n} \\ A_0(x) & A_1(x) & A_2(x) & \cdots & A_n(x) \end{pmatrix}, \qquad (4.61)$$

where the $m_{k,j}$ are modified moments

$$\begin{aligned} m_{k,j} &= \frac{1}{2\pi} \int_{-1}^{1} A_j(x) B_k(x) w(\theta) \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{1}{2\pi} \int_{0}^{\pi} (a^{i\theta}, ae^{-i\theta}; q)_j (be^{i\theta}, be^{-i\theta}; q)_k w(\theta)\, d\theta. \end{aligned}$$

Indeed, if we integrate then

$$\frac{1}{2\pi} \int_{-1}^{1} p_n(x) B_k(x) w(\theta) \frac{dx}{\sqrt{1-x^2}} = C_n \det \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,0} & m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-1,0} & m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n} \\ m_{k,0} & m_{k,1} & m_{k,2} & \cdots & m_{k,n} \end{pmatrix}$$

and this is zero when $0 \leq k \leq n-1$. If

$$D_n = \det \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,0} & m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n,0} & m_{n,1} & m_{n,2} & \cdots & m_{n,n} \end{pmatrix}, \qquad (4.62)$$

and $C_n^{-1} = (-2a)^n q^{n(n-1)/2} D_{n-1}$, then $p_n(x)$ defined in (4.61) is a monic polynomial.

4.2.2.1 Al-Salam–Chihara polynomials and little q-Laguerre polynomials

The modified moments for Al-Salam–Chihara polynomials can be computed using the integral

$$\frac{1}{2\pi}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty}\, d\theta = \frac{1}{(q;q)_\infty (ab;q)_\infty} \tag{4.63}$$

(see [31, Eq. (15.1.1)]). One easily finds

$$\begin{aligned} m_{k,j} &= \frac{1}{2\pi}\int_0^\pi (ae^{i\theta}, ae^{-i\theta}; q)_j (be^{i\theta}, be^{-i\theta}; q)_k w(\theta; a, b|q)\, d\theta \\ &= \frac{1}{2\pi}\int_0^\pi w(\theta; aq^j, bq^k|q)\, d\theta \\ &= \frac{1}{(q;q)_\infty (abq^{k+j};q)_\infty} = \frac{(ab;q)_{k+j}}{(q;q)_\infty (ab;q)_\infty}, \end{aligned}$$

so that $m_{k,j} = c_{k+j}$ and D_n given in (4.62) is a Hankel determinant. The sequence $(c_n)_{n\in\mathbb{N}}$ can be identified as the moments of a discrete measure. Indeed, by the q-binomial theorem [30, §1.3]

$$\sum_{k=0}^\infty \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1, \tag{4.64}$$

we see that (for $a = 0$)

$$\sum_{k=0}^\infty q^{kn} \frac{(ab)^k}{(q;q)_k} = \frac{1}{(abq^n;q)_\infty} = \frac{(ab;q)_n}{(ab;q)_\infty} = c_n (q;q)_\infty,$$

so that

$$c_n = \int x^n \, d\mu(x),$$

for the discrete measure μ on the q-lattice $\{q^k; k \in \mathbb{N}\}$ for which

$$\int_0^1 f(x) \, d\mu(x) = \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} f(q^k) \frac{(ab)^k}{(q;q)_k}.$$

This is the orthogonality measure for the little q-Laguerre polynomials, see (4.57). From this we have the following result:

Theorem 4.5. *Let $T_a : f \mapsto T_a(f)$ be the linear transformation that acts on polynomials as*

$$T_a(x^k) = (ae^{i\theta}, ae^{-i\theta}; q)_k.$$

Then the Al-Salam–Chihara polynomials $p_n(x; a, b|q)$ and little q-Laguerre polynomials $q_n(x; a|q)$ are connected by

$$p_n(x; a, b|q) = T_a q_n(x; ab/q|q).$$

Proof. Orthogonal polynomials are given in terms of the moments c_n of their orthogonality measure by the determinant [31, Eq. (2.1.6)]

$$p_n(x) = C_n \det \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix},$$

where C_n is a constant which fixes the normalization. If we compare this with (4.61), then we need to replace every x^k by $(ae^{i\theta}, ae^{-i\theta}; q)_k = A_k(x)$, see (4.56). The sequence $(c_n)_{n\in\mathbb{N}}$ contains the moments of the

measure

$$\mu = \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(ab)^k}{(q;q)_k} \delta_{q^k}.$$

Recall that the little q-Laguerre polynomials $q_n(x;a|q)$ satisfy the orthogonality relations (4.57), hence the orthogonal polynomials with moments $(c_n)_{n\in\mathbb{N}}$ are the little q-Laguerre polynomials $q_n(x;ab/q|q)$. Applying the transformation T_a to the determinantal expression for $q_n(x;ab/q|q)$ then shows that $T_a q_n(x;ab/q|q)$ is proportional to the Al-Salam–Chihara polynomial $p_n(x;a,b|q)$. The little q-Laguerre polynomials are given by (4.59)

$$\begin{aligned} q_n(x;a|q) &= {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q, qx \right) \\ &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(aq;q)_k (q;q)_k} q^k x^k, \end{aligned}$$

hence applying T_a to the polynomial $q_n(x;ab/q|q)$ gives

$$\begin{aligned} T_a q_n(x;ab/q|q) &= \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k}{(ab;q)_k (q;q)_k} q^k (ae^{i\theta};q)_k (ae^{-i\theta};q)_k \\ &= {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix} \middle| q, q \right), \end{aligned}$$

and this is indeed the basic hypergeometric expression (4.51) for the Al-Salam–Chihara polynomial $p_n(x;a,b|q)$. Therefore the proportionality factor is 1 and the result follows. □

The linear transformation T_a can be given explicitly and uses continuous q-Hermite polynomials $H_n(x|q)$, which are given in [37, §14.26]

[31, §13.1]. They satisfy the orthogonality

$$\frac{1}{2\pi}\int_{-1}^{1} H_n(x|q)H_m(x|q)(e^{2i\theta},e^{-2i\theta};q)_\infty \frac{dx}{\sqrt{1-x^2}} = \frac{\delta_{m,n}}{(q^{n+1};q)_\infty}, \quad (4.65)$$

and they have the generating function [37, Eq. (14.26.11)] [31, Thm. 13.1.1]

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} t^n = \frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty}, \qquad x=\cos\theta. \quad (4.66)$$

Theorem 4.6. *The linear transformation T_a that acts on polynomials as*

$$T_a x^k = (ae^{i\theta},ae^{-i\theta};q)_k$$

is given by

$$(T_a f)(x) = (ae^{i\theta},ae^{-i\theta};q)_\infty \sum_{n=0}^{\infty} f(q^n)a^n \frac{H_n(x|q)}{(q;q)_n}. \quad (4.67)$$

Proof. Obviously the transformation given in (4.67) is linear, so we only need to check that it acts properly on the monomials x^k. If we take $f(x)=x^k$ in (4.67) then

$$T_a x^k = (ae^{i\theta},ae^{-i\theta};q)_\infty \sum_{n=0}^{\infty} q^{kn}a^n \frac{H_n(x|q)}{(q;q)_n}.$$

Taking $t=aq^k$ in the generating function (4.66) gives

$$T_a x^k = \frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(aq^ke^{i\theta},aq^ke^{-i\theta};q)_\infty} = (ae^{i\theta},ae^{-i\theta};q)_k,$$

which is indeed what we need. □

The transformation T_a has an interesting isometric property, preserving certain inner products.

Proposition 4.7. *Let* $\langle f,g\rangle_{\mathrm{dis}(a)}$ *be the discrete inner product*

$$\langle f,g\rangle_{\mathrm{dis}(a)}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}f(q^n)g(q^n)\frac{a^n}{(q;q)_n}\ ,$$

and $\langle u,v\rangle_{\mathrm{cont}(a,b)}$ *be the continuous inner product*

$$\langle u,v\rangle_{\mathrm{cont}(a,b)}=\frac{1}{2\pi}\int_{-1}^{1}u(x)v(x)w(\theta;a,b|q)\frac{dx}{\sqrt{1-x^2}}\ ,$$

where $w(\theta;a,b|q)$ *is the weight function* (4.50)*. Then*

$$\langle T_af,T_bg\rangle_{\mathrm{cont}(a,b)}=\langle f,g\rangle_{\mathrm{dis}(ab)}.$$

Proof. By using (4.67) we find

$$\langle T_af,T_bg\rangle_{\mathrm{cont}(a,b)}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(q^n)g(q^m)\frac{a^nb^m}{(q;q)_n(q;q)_m}$$
$$\times\frac{1}{2\pi}\int_{-1}^{1}(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta};q)_\infty w(\theta;a,b|q)H_n(x|q)H_m(x|q)\frac{dx}{\sqrt{1-x^2}}.$$

The integral simplifies to

$$\frac{1}{2\pi}\int_{-1}^{1}(e^{2i\theta},e^{-2i\theta};q)_\infty H_n(x|q)H_m(x|q)\frac{dx}{\sqrt{1-x^2}}=\delta_{n,m}\frac{(q;q)_n}{(q;q)_\infty}\ ,$$

which follows from the orthogonality (4.65) of the continuous q-Hermite polynomials. Hence the double sum becomes a single sum and the result follows. □

As a corollary we see that the orthogonality relations of the Al-Salam–Chihara polynomials follow from the orthogonality of the little q-Laguerre polynomials. Indeed, by Theorem 4.5 we have

$T_a q_n(x; ab/q|q) = p_n(x; a, b|q)$ and by interchanging a and b we also have $T_b q_m(x; ab/q|q) = p_m(x; b, a|q)$ and the latter is equal to $p_m(x; a, b|q)$. Proposition 4.7 then shows that

$$\frac{1}{2\pi}\int_{-1}^{1} p_n(x; a, b|q)p_m(x; a, b|q)w(\theta; a, b|q)\,d\theta$$
$$= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty} q_n(q^k; ab/q|q)q_m(q^k; ab/q|q)\frac{(ab)^k}{(q;q)_k},$$

and the latter is 0 whenever $m \neq n$ by the orthogonality (4.57) of the little q-Laguerre polynomials. Clearly the norms of $p_n(x; a, b|q)$ and $q_n(x; ab/q|q)$ are also connected.

4.2.2.2 Askey–Wilson polynomials and little q-Jacobi polynomials

The modified moments $m_{k,j}$ for the Askey–Wilson weight are given by

$$\begin{aligned} m_{k,j} &= \frac{1}{2\pi}\int_0^{\pi} (ae^{i\theta}, ae^{-i\theta}; q)_j (be^{i\theta}, be^{-i\theta}; q)_k w(\theta; a, b, c, d|q)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} w(\theta; aq^j, bq^k, c, d|q)\,d\theta. \end{aligned}$$

To evaluate this integral we use the Askey–Wilson integral [31, Eq. (15.2.1)]

$$\frac{1}{2\pi}\int_0^{\pi} w(\theta; a, b, c, d|q)\,d\theta = \frac{(abcd; q)_\infty}{(q;q)_\infty (ab, ac, ad, bc, bd, cd; q)_\infty},$$
$$|a|, |b|, |c|, |d| < 1, \tag{4.68}$$

which gives

$$\begin{aligned} m_{k,j} &= \frac{(abcdq^{k+j};q)_\infty}{(q;q)_\infty (abq^{k+j}, acq^j, adq^j, bcq^k, bdq^k, cd;q)_\infty} \\ &= \frac{(abcd;q)_\infty}{(q;q)_\infty (ab,ac,ad,bc,bd,cd;q)_\infty} \\ &\times \frac{(ab;q)_{k+j}(ac;q)_j(ad;q)_j(bc;q)_k(bd;q)_k}{(abcd;q)_{k+j}}. \end{aligned} \tag{4.69}$$

This expression contains a Hankel part c_{k+j} depending only on $k+j$, with

$$c_n = c_0 \frac{(ab;q)_n}{(abcd;q)_n}, \qquad c_0 = \frac{(abcd;q)_\infty}{(q;q)_\infty (ab,ac,ad,bc,bd,cd;q)_\infty}.$$

The sequence $(c_n)_{n\in\mathbb{N}}$ can again be identified as the moments of a discrete measure on the q-lattice $\{q^k : k \in \mathbb{N}\}$. If we take $z = bq^n$ in the q-binomial theorem (4.64), then

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} b^k q^{nk} = \frac{(abq^n;q)_\infty}{(bq^n;q)_\infty} = \frac{(ab;q)_\infty}{(b;q)_\infty} \frac{(b;q)_n}{(ab;q)_n},$$

so that

$$c_n = \int_0^1 x^n \, d\nu(x), \qquad \int_0^1 f(x)\, d\nu(x) = \hat{c}_0 \sum_{k=0}^{\infty} \frac{(cd;q)_k}{(q;q)_k} (ab)^k f(q^k),$$

where $\hat{c}_0 = 1/(q;ac,ad,bc,bd,cd;q)_\infty$. This is the orthogonality measure for little q-Jacobi polynomials, see (4.58). We can now prove the following result.

Theorem 4.8. *Let $T_{a,c,d} : f \mapsto T_{a,c,d}(f)$ be the linear transformation that acts on polynomials as*

$$T_{a,c,d}(x^k) = \frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(ac;q)_k (ad;q)_k}.$$

Then the Askey–Wilson polynomials $p_n(x;a,b,c,d|q)$ *and little q-Jacobi polynomials* $q_n(x;a,b|q)$ *are connected by*

$$p_n(x;a,b,c,d|q) = T_{a,c,d} q_n(x;ab/q,cd/q|q).$$

Proof. If we insert (4.69) in the determinant (4.61), then we can take out the factor $(bc;q)_k(bd;q)_k$ in the kth row and the factor $(ac;q)_j(ad;q)_j$ in the jth column, to find

$$p_n(x;a,b,c,d|q) = C_n \prod_{k=0}^{n-1}(bc,bd;q)_k \prod_{j=0}^{n}(ac,ad;q)_j \det \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \hat{A}_0(x) & \hat{A}_1(x) & \hat{A}_2(x) & \cdots & \hat{A}_n(x) \end{pmatrix},$$

where

$$\hat{A}_k(x) = \frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(ac;q)_k(ad;q)_k}.$$

Recall that the little q-Jacobi polynomials satisfy the orthogonality (4.58)

$$\sum_{k=0}^{\infty} q_n(q^k;a,b|q) q_m(q^k;a,b|q) \frac{(bq;q)_k}{(q;q)_k}(aq)^k = 0, \tag{4.70}$$

$$m \neq n, \tag{4.71}$$

so the sequence $(c_n)_{n\in\mathbb{N}}$ contains the moments of the orthogonality measure for the little q-Jacobi polynomials $q_n(x;ab/q,cd/q|q)$. The determinant representation of the little q-Jacobi polynomials is

therefore given by

$$q_n(x;ab/q,cd/q|q) = \widehat{C}_n \det \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix},$$

where $\widehat{C}_n$ is a normalizing constant. Applying the linear transformation $T_{a,c,d}$ shows that $T_{acd}q_n(x;ab/q,cd/q|q)$ is proportional to $p_n(x;a,b,c,d|q)$. The explicit expression for the little q-Jacobi polynomials is (4.60) hence applying the transformation $T_{a,c,d}$ to it gives

$$\begin{aligned} &T_{a,c,d}q_n(x;ab/q,cd/q|q) \\ &= \sum_{k=0}^{n} \frac{(q^{-n},q)_k (abcdq^{n-1};q)_k}{(ab;q)_k (q;q)_k} q^k \frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(ac;q)_k (ad;q)_k} \\ &= {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q,q \right), \end{aligned}$$

which is indeed the basic hypergeometric expression (4.55) for the Askey–Wilson polynomial. Hence the proportionality factor is 1 and the result follows. □

The linear transformation $T_{a,c,d}$ can also be given explicitly and is in terms of the Al-Salam–Chihara polynomials. We will use a multiple of the polynomials $p_n(x;a,b|q)$ defined above and put

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n} p_n(x;a,b|q) = \frac{(ab;q)_n}{a^n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix} \middle| q,q \right).$$

They have the following generating function ([37, Eq. (14.8.13)] [31, Eq. (15.1.10)]

$$\sum_{n=0}^{\infty} \frac{Q_n(x;a,b|q)}{(q;q)_n} t^n = \frac{(at,bt;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}, \qquad x=\cos\theta. \tag{4.72}$$

Theorem 4.9. *The linear transformation $T_{a,c,d}$ that acts on polynomials as*

$$T_{a,c,d}x^k = \frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(ac;q)_k(ad;q)_k}$$

is given by

$$(T_{a,c,d}f)(x) = \frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(ac,ad;q)_\infty} \sum_{n=0}^{\infty} f(q^n)a^n \frac{Q_n(x;c,d|q)}{(q;q)_n}. \tag{4.73}$$

Proof. The transformation in (4.73) is obviously linear, so we need only to check how it acts on polynomials x^k. Taking $f(x)=x^k$ in (4.73) gives

$$T_{a,c,d}x^k = \frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(ac,ad;q)_\infty} \sum_{n=0}^{\infty} q^{kn}a^n \frac{Q_n(x;c,d|q)}{(q;q)_n},$$

and if we put $t=aq^k$ in the generating function (4.72), then this gives

$$T_{a,c,d}x^k = \frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(ac,ad;q)_\infty} \frac{(acq^k,adq^k;q)_\infty}{(aq^ke^{i\theta},aq^ke^{-i\theta};q)_\infty} = \frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(ac,ad;q)_k},$$

which is the desired result. □

This transformation was given explicitly in [32], see e.g. their equation (4.5). Their formula (4.7) also gives the Askey–Wilson polynomials as the image of applying $T_{a,c,d}$ to little q-Jacobi polynomials. When $c=d=0$ the transformation $T_{a,c,d}$ is equal to T_a, which

reflects the fact that $Q_n(x;0,0|q) = H_n(x|q)$. This transformation $T_{a,c,d}$ also obeys a Plancherel type result in the following sense.

Proposition 4.10. *Let* $\langle f,g\rangle_{\mathrm{dis}(a,b)}$ *be the discrete inner product*

$$\langle f,g\rangle_{\mathrm{dis}(a,b)} = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} f(q^n)g(q^n)a^n\frac{(b;q)_n}{(q;q)_n},$$

and $\langle u,v\rangle_{\mathrm{cont}(a,b,c,d)}$ *be the continuous inner product*

$$\langle u,v\rangle_{\mathrm{cont}(a,b,c,d)} = \frac{(ac,ad,bc,bd,cd;q)_\infty}{2\pi}\int_{-1}^{1} u(x)v(x)w(\theta;a,b,c,d|q)\frac{dx}{\sqrt{1-x^2}},$$

where $w(\theta;a,b,c,d|q)$ *is the Askey–Wilson weight function* (4.54). *Then*

$$\langle T_{a,c,d}f, T_{b,c,d}g\rangle_{\mathrm{cont}(a,b,c,d)} = \langle f,g\rangle_{\mathrm{dis}(ab,cd)}.$$

Proof. If we use the expression (4.73) for the transformation $T_{a,c,d}$ then

$$\langle T_{a,c,d}f, T_{b,c,d}g\rangle_{a,b,c,d} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(q^n)g(q^m)\frac{a^nb^m}{(q;q)_n(q;q)_m(ac,ad,bc,bd;q)_\infty}$$
$$\times\frac{1}{2\pi}\int_0^{\pi}(ae^{i\theta},ae^{-i\theta};q)_\infty(be^{i\theta},be^{-i\theta};q)_\infty Q_n(x;c,d|q)Q_m(x;c;d|q)w(\theta;a,b,c,d)\,d\theta.$$

Observe that the integral in this expression is

$$\frac{1}{2\pi}\int_0^{\pi} Q_n(x;c,d|q)Q_m(x;c,d|q)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\,d\theta$$
$$= \frac{\delta_{n,m}}{(q^{n+1};q)_\infty(cdq^n;q)_\infty},$$

(see [37, Eq. (14.8.2)]), so that the double sum becomes a single sum

$$\frac{1}{(q;q)_\infty(ac,ad,bc,bd,cd;q)_\infty}\sum_{n=0}^{\infty} f(q^n)g(q^n)(ab)^n\frac{(cd;q)_n}{(q;q)_\infty},$$

which is the desired expression in terms of the discrete inner product $\langle f, g\rangle_{\mathrm{dis}(ab,cd)}$. □

As a corollary, the orthogonality for the Askey–Wilson polynomials now follows from the orthogonality of the little q-Jacobi polynomials. Indeed, if we use Theorem 4.8 then $T_{a,c,d}q_n(x; ab/q, cd/q|q) = p_n(x; a, b, c, d|q)$ and $T_{b,c,d}q_m(x; ab/q, cd/q|q) = p_m(x; b, a, c, d|q)$ and the latter is equal to $p_m(x; a, b, c, d|q)$. Hence by Proposition 4.7

$$\frac{(ac, ad, bc, bd, cd; q)_\infty}{2\pi} \int_0^\pi p_n(x; a, b, c, d|q) p_m(x; a, b, c, d|q) w(\theta; a, b, c, d|q)\, d\theta$$
$$= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} q_n(q^k; ab/q, cd/q|q) q_m(q^k; ab/q, cd/q|q) (ab)^k \frac{(cd; q)_k}{(q; q)_k},$$

and this is 0 whenever $m \neq n$ because of the orthogonality relations (4.58) for the little q-Jacobi polynomials. Clearly one can also relate the norms of $p_n(x; a, b, c, d|q)$ with those of $q_n(x; ab/q, cd/q|q)$ by putting $n = m$.

4.2.2.3 Continuous dual q-Hahn polynomials and little q-Laguerre polynomials

The modified moments for the continuous dual q-Hahn polynomials correspond to the modified moments of the Askey–Wilson weight with $d = 0$,

$$\begin{aligned} m_{k,j} &= \frac{1}{2\pi} \int_0^\pi (ae^{i\theta}, ae^{-i\theta}; q)_j ((be^{i\theta}, be^{-i\theta}; q)_k w(\theta|a, b, c|q)\, d\theta \\ &= \frac{(ab; q)_{k+j} (ac; q)_j (bc; q)_k}{(q; q)_\infty (ab, ac, bc; q)_\infty}. \end{aligned} \tag{4.74}$$

The Hankel part $(ab;q)_{k+j}$ corresponds to the moments of the discrete measure μ that we used in Section 4.2.2.1, which is the orthogonality measure for the little q-Laguerre polynomials. We can then prove the following result

Theorem 4.11. *Let $T_{a,c} : f \mapsto T_{a,c}f$ be the linear transformation that acts on polynomials as*

$$T_{a,c}x^k = \frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(ac;q)_k}.$$

Then the continuous dual q-Hahn polynomials $p_n(x;a,b,c|q)$ and the little q-Laguerre polynomials $q_n(x;a|q)$ are connected by

$$p_n(x;a,b,c|q) = T_{a,c}q_n(x;ab/q|q).$$

Proof. If we insert the modified moments (4.74) in (4.61), then we can take out the factor $(bc;q)_k$ in row k and the factor $(ac;q)_j$ in column j. This gives

$$p_n(x;a,b,c|q) = C_n \prod_{k=0}^{n-1}(bc;q)_k \prod_{j=0}^{n}(ac;q)_j \det\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \tilde{A}_0(x) & \tilde{A}_1(x) & \tilde{A}_2(x) & \cdots & \tilde{A}_n(x) \end{pmatrix},$$

where

$$\tilde{A}_k(x) = \frac{(ae^{i\theta}, ae^{-i\theta};q)_k}{(ac;q)_k},$$

and $(c_n)_{n\in\mathbb{N}}$ are the moments of the measure μ, which is the measure for the little q-Laguerre polynomials $q_n(x;ab/q|q)$. Hence applying $T_{a,c}$ to $q_n(x;ab/q|q)$ gives the continuous dual q-Hahn polynomials. □

The linear transformation $T_{a,c}$ can again be given explicitly and is in term of the continuous big q-Hermite polynomials $H_n(x;a|q)$, which

are given in [37, §14.18]. The orthogonality relations are

$$\frac{1}{2\pi}\int_{-1}^{1} H_n(x;a|q)H_m(x;a|q)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}\frac{dx}{\sqrt{1-x^2}}=\frac{\delta_{m,n}}{(q^{n+1},q)_\infty},$$

and they have a generating function

$$\sum_{n=0}^{\infty}\frac{H_n(x;a|q)}{(q;q)_n}t^n=\frac{(at;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty},\qquad x=\cos\theta. \tag{4.75}$$

Observe that for $a=0$ they reduce to the continuous q-Hermite polynomials that we used in Section 4.2.2.1.

Theorem 4.12. *The linear transformation $T_{a,c}$ that acts on polynomials as*

$$T_{a,c}x^k=\frac{(ae^{i\theta},ae^{-i\theta};q)_k}{(ac;q)_k}$$

is given by

$$(T_{a,c}f)(x)=\frac{(ae^{i\theta},ae^{-i\theta};q)_\infty}{(ac;q)_\infty}\sum_{n=0}^{\infty}f(q^n)a^n\frac{H_n(x;c|q)}{(q;q)_n}. \tag{4.76}$$

Proof. The linearity is obvious and the action of $T_{a,c}$ on x^k can easily be checked, using the generating function (4.75). □

Proposition 4.13. *Let $\langle f,g\rangle_{\mathrm{dis}(a)}$ be the discrete inner product*

$$\langle f,g\rangle_{\mathrm{dis}(a)}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}f(q^n)g(q^n)\frac{a^n}{(q;q)_n},$$

and $\langle u,v\rangle_{\mathrm{cont}(a,b,c)}$ the continuous inner product

$$\frac{(ac,bc;q)_\infty}{2\pi}\int_{-1}^{1}u(x)v(x)w(\theta;a,b,c|q)\,d\theta,$$

where $w(\theta;a,b,c|q)$ *is the weight function* (4.52)*. Then*

$$\langle T_{a,c}f, T_{b,c}g\rangle_{\mathrm{cont}(a,b,c)} = \langle f,g\rangle_{\mathrm{dis}(ab)}.$$

Proof. This follows from Proposition 4.10 by taking $d=0$, or from the orthogonality of the continuous big q-Hermite polynomials in a similar way as in the proofs of Proposition 4.7 and 4.10. □

As a corollary one can deduce the orthogonality of the continuous dual q-Hahn polynomials from the orthogonality of the little q-Laguerre polynomials, by using Theorem 4.11.

It is interesting to see how these various families of basic hypergeometric polynomials are connected, see Figure 4.1. Continuous q-Hermite polynomials are at the bottom of the q-Askey scheme of basic hypergeometric polynomials [37, p. 413], with a weight function $w(\theta|q) = (e^{2i\theta}, e^{-2i\theta};q)_\infty$. You need them for the transformation T_a that maps little q-Laguerre polynomials to Al-Salam–Chihara polynomials, which have a weight $w(\theta;a,b|q)$ with two parameters a,b. A more general transformation $T_{a,c}$ maps the same little q-Laguerre polynomials to continuous dual q-Hahn polynomials, which have a weight function $w(\theta;a,b,c|q)$ with three parameters. The Al-Salam–Chihara polynomials in turn are needed for the transformation $T_{a,c,d}$ which maps little q-Jacobi polynomials to Askey–Wilson polynomials which have a weight function $w(\theta;a,b,c,d|q)$ with four parameters. These transformations T_a, $T_{a,c}$ and $T_{a,c,d}$ seem to be special cases of the Askey–Wilson function transform given in [39, Eq. (5.9)] and [57, p. 312], since the polynomials $H_n(x|q)$, $H_n(x;a|q)$ and $Q_n(x;a,b|q)$ in

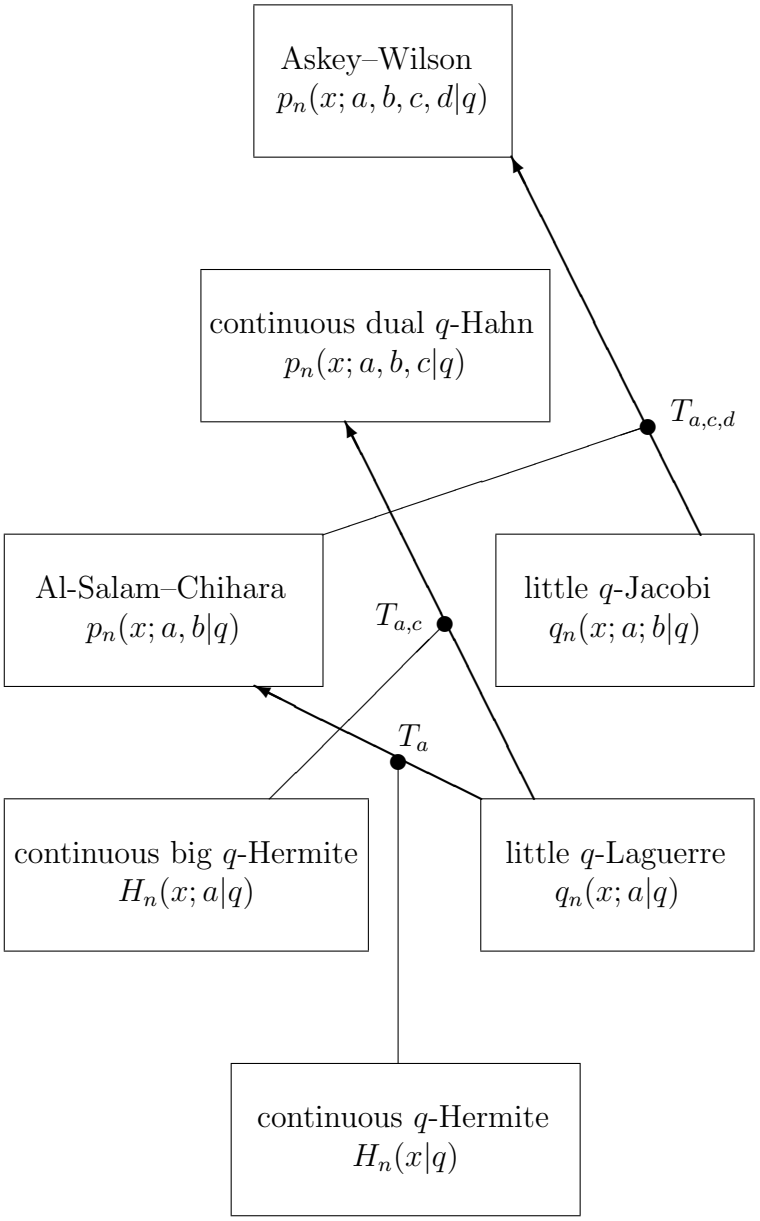


FIGURE 4.1: Schematic presentation of the hypergeometric orthogonal polynomials and transformations.

our tranformations $T_a, T_{a,c}$ and $T_{a,c,d}$ are special cases of the Askey–Wilson polynomial $p_n(x;a,b,c,d|q)$, but this needs a little more inspection of those papers. Our Plancherel type formulas then correspond to [39, Thm. 1 and Prop. 3] and [57, Thm. 5.1].

4.2.3 Multiple basic hypergeometric polynomials

We will now use the results from the previous section to construct multiple hypergeometric orthogonal polynomials for Askey–Wilson weights, continuous dual q-Hahn weights and Al-Salam–Chihara weights from discrete multiple orthogonal polynomials on the q-lattice $\{q^k, k \in \mathbb{N}\}$. We always use r weights $(w_1, \ldots, w_r)$ obtained by changing the parameter a to a vector $\vec{a} = (a_1, \ldots, a_r)$. We start by recalling some results for the multiple little q-Laguerre and the multiple little q-Jacobi polynomials.

4.2.3.1 Little q-Laguerre polynomials

The little q-Laguerre polynomials are given by [37, §14.20]

$$q_n(x; a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q, qx\right) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k (aq; q)_k} q^k x^k. \quad (4.77)$$

When taking $a = q^\alpha$, they can be obtained by the Rodrigues formula

$$(qx; q)_\infty x^\alpha q_n(x; a|q) = \frac{q^{\binom{n}{2}} a^n (1-q)^n}{(aq; q)_n} D_p^n x^{n+\alpha} (qx; q)_\infty, \quad (4.78)$$

where $p = 1/q$ and D_p is the p-difference operator

$$D_p f(x) = \frac{f(px) - f(x)}{x(p-1)} = \frac{q}{1-q} \frac{f(px) - f(x)}{x}.$$

Note that these polynomials are neither monic nor orthonormal, but they are normalized by $q_n(0; a|q) = 1$.

Multiple little q-Laguerre polynomials $q_{\vec{n}}(x; \vec{a}|q)$ for the multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ depend on r parameters $\vec{a} = (a_1, a_2, \ldots, a_r)$ and

satisfy multiple orthogonality relations

$$\sum_{k=0}^{\infty} q_{\vec{n}}(q^k;\vec{a}|q)q^{k\ell}\frac{(a_jq)^k}{(q;q)_k}=0, \qquad 0\le k\le n_j-1, \tag{4.79}$$

for $1\le j\le r$. They can be defined by the Rodrigues formula

$$(qx;q)_\infty q_{\vec{n}}(x;\vec{a}|q)=C_{\vec{n}}(\vec{a})\prod_{j=1}^{r}\left(x^{-\alpha_j}D_p^{n_j}x^{n_j+\alpha_j}\right)\ (qx;q)_\infty, \tag{4.80}$$

where $a_j=q^{\alpha_j}$ and the p-difference operators $x^{-\alpha_j}D_p^{n_j}x^{n_j+\alpha_j}$ are commuting.

Theorem 4.14. *If we take the normalizing factor in* (4.80) *as*

$$C_{\vec{n}}(\vec{a})=(1-q)^{|\vec{n}|}\prod_{j=1}^{r}\frac{q^{\binom{n_j}{2}}a_j^{n_j}}{(a_jq;q)_{n_j}}, \tag{4.81}$$

then an explicit expression for the multiple little q-Laguerre polynomials is given by

$$q_{\vec{n}}(x;\vec{a}|q)=\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\prod_{j=1}^{r}\frac{(q^{-n_j};q)_{k_j}}{(q;q)_{k_j}(a_jq;q)_{k_j}}\prod_{j=1}^{r-1}\frac{(a_jq^{n_j+1};q)_{k_{j+1}+\cdots+k_r}}{(a_jq^{k_j+1};q)_{k_{j+1}+\cdots+k_r}}\frac{(qx)^{|\vec{k}|}}{q^{\sum_{j=1}^{r}n_j\sum_{i=j+1}^{r}k_i}}. \tag{4.82}$$

These multiple little q-Laguerre polynomials are normalized so that $q_{\vec{n}}(0;\vec{a}|q)=1$.

Proof. We will use induction on r. For $r=1$ we have the result (4.77) for the usual little q-Laguerre polynomials in [37, §14.20] and

$$C_n(a)=\frac{q^{\binom{n}{2}}a^n(1-q)^n}{(aq;q)_n}, \tag{4.83}$$

as can be deduced from (4.78).

Suppose the result is true for $r-1$. The difference operators $x^{-\alpha_j} D_p^{n_j} x^{n_j+\alpha_j}$, $1 \leq j \leq r$ are all commuting, so the order in which we take the product of these operators is irrelevant. The Rodrigues formula (4.80) can then be written as

$$(qx;q)_\infty q_{\vec{n}}(x;\vec{a}|q) = C_{\vec{n}}(\vec{a}) x^{-\alpha_1} D_p^{n_1} x^{n_1+\alpha_1} \prod_{j=2}^{r} \left(x^{-\alpha_j} D_p^{n_j} x^{n_j+\alpha_j} \right) (qx;q)_\infty.$$

By the induction hypothesis, the product $\prod_{j=2}^{r}$ can be written as an $(r-1)$-fold sum and we have

$$\begin{aligned}
&(qx;q)_\infty q_{\vec{n}}(x;\vec{a}|q) = \\
&\frac{C_{\vec{n}}(\vec{a})}{C_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{e}_1)} x^{-\alpha_1} D_p^{n_1} x^{n_1+\alpha_1} (qx;q)_\infty q_{\vec{n}-n_1\vec{e}_1}(x;\vec{a}-a_1\vec{e}_1|q) \\
&= \frac{C_{\vec{n}}(\vec{a})}{C_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{e}_1)} \sum_{k_2=0}^{n_2} \cdots \sum_{k_r=0}^{n_r} \prod_{j=2}^{r} \frac{(q^{-n_j};q)_{k_j}}{(q;q)_{k_j}(a_j q;q)_{k_j}} \\
&\quad \times \prod_{j=2}^{r-2} \frac{a_j q^{n_j+1};q)_{k_{j+1}+\cdots+k_r}}{(a_j q^{k_j+1};q)_{k_{j+1}+\cdots+k_r}} \frac{(qx)^{k_2+\cdots+k_r}}{q^{\sum_{j=2}^{r} n_j \sum_{i=j+1}^{r} k_i}} \\
&\quad \times x^{-\alpha_1} D_p^{n_1} x^{n_1+\alpha_1+k_2+\cdots+k_r} (qx;q)_\infty.
\end{aligned}$$

The p-difference on the last line can be worked out using the Rodrigues formula (4.78) for $r=1$

$$x^{-\alpha_1} D_p^{n_1} x^{n_1+\alpha_1+k_2+\cdots+k_r} (qx;q)_\infty = \frac{(qx;q)_\infty}{C_{n_1}(a_1 q^{k_2+\cdots+k_r})} q_{n_1}(x;a_1 q^{k_2+\cdots+k_r}|q),$$

and if we use the sum (4.77) and the expression (4.83) for $C_{n_1}(a_1 q^{k_2+\cdots+k_r})$ then after some calculus we find the desired expression (4.82), provided

$$C_{\vec{n}}(\vec{a}) = \frac{q^{\binom{n_1}{2}} a_1^{n_1} (1-q)^{n_1}}{(a_1 q;q)_{n_1}} C_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{e}_1).$$

This is achieved by taking $C_{\vec{n}}(\vec{a})$ as in (4.81). □

4.2.3.2 Little q-Jacobi polynomials

Multiple little q-Jacobi polynomials were introduced in [53], where two kinds were given. Here we only deal with the multiple little q-Jacobi polynomials of the first kind and we will use a different normalization. Recall that the little q-Jacobi polynomials are given by

$$q_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\middle|\, q, qx\right) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (abq^{n+1};q)_k}{(q;q)_k (aq;q)_k} q^k x^k, \tag{4.84}$$

where $a = q^\alpha$ and $b = q^\beta$. They are given by the Rodrigues formula

$$\frac{(qx;q)_\infty}{(bqx;q)_\infty} x^\alpha q_n(x;a,b|q) = C_n(a,b) D_p^n x^{n+\alpha} \frac{(qx;q)_\infty}{(bq^{n+1}x;q)_\infty}, \tag{4.85}$$

where

$$C_n(a,b) = C_n(a) = \frac{q^{\binom{n}{2}} a^n (1-q)^n}{(aq;q)_n}. \tag{4.86}$$

These little q-Jacobi polynomials have the orthogonality relations

$$\sum_{k=0}^{\infty} q_n(q^k;a,b|q) q_m(q^k;a,b|q) q^k \frac{(bq;q)_k}{(q;q)_k} a^k = 0, \qquad n \neq m.$$

Observe that for $b = 0$ we retrieve the little q-Laguerre polynomials.

Multiple little q-Jacobi polynomials (of the first kind) $q_{\vec{n}}(x;\vec{a},b|q)$ are obtained by changing the parameter a to a vector $\vec{a} = (a_1,\ldots,a_r)$. If $\vec{n} = (n_1,n_2,\ldots,n_r)$ the orthogonality relations then become

$$\sum_{k=0}^{\infty} q_{\vec{n}}(q^k;\vec{a},b|q) q^{k\ell} q^k \frac{(bq;q)_k}{(q;q)_k} a_j^k = 0, \qquad 0 \leq \ell \leq n_j - 1, \tag{4.87}$$

for $1 \leq j \leq r$. One needs the condition $\alpha_i - \alpha_j \notin \mathbb{Z}$ in order that these orthogonality relations determine the multiple orthogonal

polynomials in a unique way. In [53, Thm. 2.2] a Rodrigues formula was given

$$\frac{(qx;q)_\infty}{(bqx;q)_\infty} q_{\vec{n}}(x;\vec{a},b|q) = C_{\vec{n}}(\vec{a},b) \prod_{j=1}^{r} \left(x^{-\alpha_j} D_p^{n_j} x^{n_j+\alpha_j} \right) \frac{(qx;q)_\infty}{(bq^{|\vec{n}|+1}x;q)_\infty}, \tag{4.88}$$

where $C_{\vec{n}}(\vec{a},b)$ is a constant that determines the normalization. An expression in terms of a generalized basic hypergeometric function was given in [53, Eq. (2.7)]

$$\frac{(qx;q)_\infty}{(bqx;q)_\infty} q_{\vec{n}}(x;\vec{a},b|q) = {}_{r+1}\phi_r \left(\begin{matrix} q^{-|\vec{n}|}/b, a_1 q^{n_1+1}, \ldots, a_r q^{n_r+1} \\ a_1 q, a_2 q, \ldots, a_r q \end{matrix} \middle| q, bqx \right).$$

Here we used the normalization so that $q_{\vec{n}}(0;\vec{a},b|q) = 1$, which is different from the normalization in [53]. Note that the limit $b \to 0$ gives the multiple little q-Laguerre polynomials $q_{\vec{n}}(x;\vec{a}|q)$, so that we get the generalized hypergeometric representation

$$(qx;q)_\infty q_{\vec{n}}(x;\vec{a}|q) = {}_r\phi_r \left(\begin{matrix} a_1 q^{n_1+1}, \ldots, a_r q^{n_r+1} \\ a_1 q, a_2 q, \ldots, a_r q \end{matrix} \middle| q, q^{-|\vec{n}|+1} x \right).$$

An expression for the multiple little q-Jacobi polynomials in terms of a finite r-fold sum is given by

Theorem 4.15. *If we take the normalizing factor in* (4.88) *as* $C_{\vec{n}}(\vec{a};b) = C_{\vec{n}}(a)$ *as in* (4.81)*, then an explicit expression for the multiple little*

q-Jacobi polynomials (of the first kind) is given by

$$q_{\vec{n}}(x;\vec{a},b|q) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\prod_{j=1}^{r}\frac{(q^{-n_j};q)_{k_j}(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{\sum_{i=j}^{r}k_i}}{(q;q)_{k_j}(a_jq;q)_{\sum_{i=j}^{r}k_i}}$$
$$\times\prod_{j=1}^{r-1}\frac{(a_jq^{n_j+1};q)_{k_{j+1}+\cdots+k_r}}{(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{k_{j+1}+\cdots+k_r}}\frac{(qx)^{|\vec{k}|}}{q^{\sum_{j=1}^{r}n_j\sum_{i=j+1}^{r}k_i}}. \quad (4.89)$$

These multiple little q-Jacobi polynomials are normalized so that $q_{\vec{n}}(0;\vec{a},b|q)=1$.

Note that for $b=0$ we get the expression (4.82) in Theorem 4.14.

Proof. The proof is again by induction on r. For $r=1$ one has the usual little q-Jacobi polynomials in (4.84) with $C_n(a,b)$ given in (4.86). Observe that this normalizing factor is independent of b.

Suppose that the result holds for $r-1$. Then the Rodrigues formula (4.88) gives

$$\frac{(qx;q)_\infty}{(bqx;q)_\infty}q_{\vec{n}}(x;\vec{a},b|q) =$$
$$C_{\vec{n}}(\vec{a},b)x^{-\alpha_1}D_p^{n_1}x^{n_1+\alpha_1}\prod_{j=2}^{r}x^{-\alpha_j}D_p^{n_j}x^{n_j+\alpha_j}\frac{(qx;q)_\infty}{(bq^{|\vec{n}|+1};q)_\infty}$$
$$=\frac{C_{\vec{n}}(\vec{a},b)}{C_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{a},b)}x^{-\alpha_1}D_p^{n_1}x^{n_1+\alpha_1}q_{\vec{n}-n_1\vec{e}_1}(x;\vec{a}-a_1\vec{e}_1;bq^{n_1}|q),$$

where we used the Rodrigues formula (4.88) with $r-1$ for the product $\prod_{j=2}^{r}$. Now use the induction hypothesis to express $q_{\vec{n}-n_1\vec{e}_1}(x;\vec{a}-$

$a_1\vec{e}_1, bq^{n_1}|q)$ as an $(r-1)$-fold sum to find

$$q_{\vec{n}}(x;\vec{a},b|q) =$$

$$\frac{C_{\vec{n}}(\vec{a},b)}{C_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{a},b)}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\prod_{j=2}^{r}\frac{(q^{-n_j};q)_{k_j}(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{\sum_{i=j}^{r}k_i}}{(q;q)_{k_j}(aq;q)_{\sum_{i=j}^{r}k_i}}$$

$$\times\prod_{j=2}^{r-1}\frac{(a_jq^{n_j+1};q)_{\sum_{i=j+1}^{r}k_i}}{(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{\sum_{i=j+1}^{r}k_i}}\frac{q^{k_2+\cdots+k_r}}{q^{\sum_{j=2}^{r}n_j\sum_{i=j+1}^{r}k_i}}x^{-\alpha_1}D_p^{n_1}x^{n_1+\alpha_1+k_2+\cdots+k_r}\frac{(qx;q)_\infty}{(bq^{n_1+1};q)_\infty}.$$

Now use the Rodrigues formula (4.85) for $r=1$ to find

$$D_p^{n_1}x^{n_1+\alpha_1+k_2+\cdots+k_r}\frac{(qx;q)_\infty}{(bq^{n_1+1};q)_\infty}$$

$$=\frac{1}{C_{n_1}(a_1q^{k_2+\cdots+k_r},bq^{n_1})}\frac{(qx;q)_\infty}{(bqx;q)_\infty}x^{\alpha_1+k_2+\cdots+k_r}q_{n_1}(x;aq^{k_2+\cdots+k_r},b|q).$$

Then use the sum (4.84) and (4.86) to find

$$q_{\vec{n}}(x;\vec{a},b|q)=\frac{C_{\vec{n}}(\vec{a},b)}{q^{\binom{n_1}{2}}a_1^n(1-q)^nC_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{a},b)}$$

$$\times\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\prod_{j=1}^{r}\frac{(q^{-n_j};q)_{k_j}}{(q;q)_{k_j}}\prod_{j=2}^{r}\frac{(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{\sum_{i=j}^{r}k_i}}{(a_jq;q)_{\sum_{i=j}^{r}k_i}}$$

$$\times\prod_{j=2}^{r-1}\frac{(a_jq^{n_j+1};q)_{\sum_{i=j+1}^{r}k_i}}{(a_jbq^{\sum_{i=1}^{j}n_i+1};q)_{\sum_{i=j+1}^{r}k_i}}\frac{(a_1q^{k_2+\cdots+k_r+1};q)_{n_1}(a_1bq^{k_2+\cdots+k_r+n_1+1};q)_{k_1}}{(a_1q^{k_2+\cdots+k_r+1};q)_{k_1}}\frac{(qx)^{|\vec{k}|}}{q^{\sum_{j=1}^{r}n_j\sum_{i=j+1}^{r}k_i}}.$$

Now use

$$\begin{aligned}(a_1q^{k_2+\cdots+k_r};q)_{n_1} &= \frac{(a_1q;q)_{n_1}(a_1q^{n_1+1};q)_{k_2+\cdots+k_r}}{(a_1q;q)_{k_2+\cdots+k_r}},\\ (a_1q^{k_2+\cdots+k_r};q)_{k_1} &= \frac{a_1q;q)_{k_1+\cdots+k_r}}{(a_1q;q)_{k_2+\cdots+k_r}},\\ (a_1bq^{k_2+\cdots+k_r+n_1+1};q)_{k_1} &= \frac{a_1bq^{n_1+1};q)_{k_1+\cdots+k_r}}{(a_1bq^{n_1+1};q)_{k_2+\cdots+k_r}}\end{aligned}$$

to find the r-fold sum in (4.89). The normalization $q_{\vec{n}}(0;\vec{a},b|q)=1$ is obtained when

$$\frac{C_{\vec{n}}(\vec{a},b)(a_1q;q)_{n_1}}{q^{\binom{n_1}{2}}a_1^n(1-q)^nC_{\vec{n}-n_1\vec{e}_1}(\vec{a}-a_1\vec{a},b)}=1,$$

which holds for $C_{\vec{n}}(\vec{a},b)=C_{\vec{n}}(\vec{a})$ in (4.81). Observe that this factor does not depend on b. □

4.2.3.3 Multiple Al-Salam–Chihara polynomials

In this section we will take r weights $(w_1,w_2,\ldots,w_r)$ on $[-1,1]$ with $w_r(\theta)=w(\theta;a_j,b|q)$, where $w(\theta;a,b|q)$ is the Al-Salam–Chihara weight given in (4.50) and $\vec{a}=(a_1,\ldots,a_r)$ such that $a_j=q^{\alpha_j}$ and $\alpha_i-\alpha_j\notin\mathbb{Z}$ whenever $i\neq j$. As usual with the Al-Salam–Chihara weight, we take $|a_j|<1$ and $|b|<1$. The corresponding multiple orthogonal polynomials can then be obtained from the little q-Laguerre polynomials by using Proposition 4.7 and the transformation T_b given in Theorem 4.6. In fact Theorem 4.5 can be extended to multiple orthogonal polynomials as follows.

Theorem 4.16. *The multiple Al-Salam–Chihara polynomials $p_{\vec{n}}(x;\vec{a},b|q)$ for the weights $w(\theta;a_1,b|q),w(\theta;a_2,b|q),\ldots,w(\theta;a_r,b|q)$, with $w(\theta;a,b|q)$ given in (4.50), are given by $p_{\vec{n}}(x;\vec{a},b|q)=T_bq_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)$, where T_b is the linear transformation given in Theorem 4.6 and $q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)$ are*

the multiple little q-Laguerre polynomials given in Section 4.2.3.1. An explicit expression is given by

$$p_{\vec{n}}(x;\vec{a},b|q) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\prod_{j=1}^{r}\frac{(q^{-n_j};q)_{k_j}}{(q;q)_{k_j}(a_jb;q)_{k_j}}\prod_{j=1}^{r-1}\frac{(a_jbq^{n_j};q)_{k_{j+1}+\cdots+k_r}}{(a_jbq^{k_j};q)_{k_{j+1}+\cdots+k_r}} \times \frac{q^{|\vec{k}|}}{q^{\sum_{j=1}^{r}n_j\sum_{i=j+1}^{r}k_i}}(be^{i\theta},be^{-i\theta};q)_{|\vec{k}|}.$$

Proof. Let T_b be the linear tarnsformation that acts on polynomials like

$$T_bx^k = (be^{i\theta},be^{-i\theta};q)_k,$$

then we apply this to the multiple little q-Laguerre polynomials with parameters $\frac{b}{q}\vec{a} = \left(\frac{a_1b}{q},\ldots,\frac{a_rb}{q}\right)$ to find $T_bq_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)$. If we use Proposition 4.7 then

$$\langle T_{a_j}x^\ell, T_bq_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)\rangle_{\mathrm{cont}(a_j,b)} = \langle x^\ell, q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)\rangle_{\mathrm{dis}(a_jb)},$$

which is

$$\frac{1}{2\pi}\int_{-1}^{1}(a_je^{i\theta},a_je^{-i\theta};q)_\ell T_bq_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)w(\theta;a_j,b|q)\,\frac{dx}{\sqrt{1-x^2}} = \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}q^{k\ell}q_{\vec{n}}(q^k;\frac{b}{q}\vec{a}|q)\frac{(a_jb)^k}{(q;q)_k}.$$

The latter sum is 0 because of (4.79) when $0 \le \ell \le n_j - 1$ for $1 \le j \le r$. This shows that $T_bq_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)$ satisfies the multiple orthogonality conditions with respect to the r Al-Salam–Chihara weights $w(\theta;a_1,b|q), w(\theta;a_2,b|q),\ldots,w(\theta;a_r,b|q)$. The explicit expression as an r-fold sum is obtained by applying T_b to the sum (4.82) in Theorem 4.14 for the parameters ba_j/q. □

4.2.3.4 Multiple continuous dual q-Hahn polynomials

We now take r weights $(w_1, \dots, w_r)$ on $[-1, 1]$ by using the continuous dual q-Hahn weight $w(\theta; a_j, b, c|q)$ of (4.52) with r different parameters $\vec{a} = (a_1, \dots, a_r)$, keeping b, c fixed. Again we let $a_j = q^{\alpha_j}$ and assume that $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. This ensures that the multiple orthogonality conditions

$$\int_{-1}^{1} p_{\vec{n}}(x; \vec{a}, b, c|q) x^{\ell} w(\theta; a_j, b, c|q) \frac{dx}{\sqrt{1-x^2}} = 0, \qquad 0 \leq \ell \leq n_j - 1,$$

for $1 \leq j \leq r$, give $|\vec{n}|$ equations that determine the $p_{\vec{n}}(x; \vec{a}, b, c|q)$ uniquely (up to a multiplicative factor). These multiple continuous dual q-Hahn polynomials can be obtained by using the linear transformation $T_{b,c}$ given in Theorem 4.12 to the multiple little q-Laguerre polynomials. The extension of Theorem 4.11 to multiple continuous dual q-Hahn polynomials is:

Theorem 4.17. *The multiple continuous dual q-Hahn polynomials $p_{\vec{n}}(x; \vec{a}, b, c|q)$ for the weights $w(\theta; a_1, b, c|q), w(\theta; a_2, b, c|q), \dots, w(\theta; a_r, b, c|q)$, with $w(\theta; a, b, c|q)$ given in (4.52), are given by $p_{\vec{n}}(x; \vec{a}, b, c|q) = T_{b,c} q_{\vec{n}}(x; \frac{b}{q}\vec{a}|q)$, where $T_{b,c}$ is the linear transformation given in Theorem 4.12 and $q_{\vec{n}}(x; \frac{b}{q}\vec{a}|q)$ are the multiple little q-Laguerre polynomials given in Section 4.2.3.1. An explicit expression is given by*

$$p_{\vec{n}}(x; \vec{a}, b, c|q) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \prod_{j=1}^{r} \frac{(q^{-n_j}; q)_{k_j}}{(q; q)_{k_j} (a_j b; q)_{k_j}} \prod_{j=1}^{r-1} \frac{(a_j b q^{n_j}; q)_{k_{j+1}+\cdots+k_r}}{(a_j b q^{k_j}; q)_{k_{j+1}+\cdots+k_r}}$$
$$\times \frac{q^{|\vec{k}|}}{q^{\sum_{j=1}^{r} n_j \sum_{i=j+1}^{r} k_i}} \frac{(be^{i\theta}, be^{-i\theta}; q)_{|\vec{k}|}}{(bc; q)_{|\vec{k}|}}.$$

Proof. If we use Proposition 4.13 then

$$\langle T_{a_j,c}x^\ell, T_{b,c}q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)\rangle_{\mathrm{cont}(a_j,b,c)} = \langle x^\ell, q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)\rangle_{\mathrm{dis}(a_j b)},$$

which is

$$\frac{(a_jc,bc;q)_\infty}{2\pi}\int_{-1}^{1}\frac{(a_je^{i\theta},a_je^{-i\theta};q)_\ell}{(a_jc;q)_k}T_{b,c}q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)w(\theta;a_j,b,c|q)\,\frac{dx}{\sqrt{1-x^2}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}q^{k\ell}q_{\vec{n}}(q^k;\frac{b}{q}\vec{a}|q)\frac{(a_jb)^k}{(q;q)_k}$$

and the latter sum is 0 whenever $0\le\ell\le n_j-1$ because of the multiple orthogonality conditions (4.79) for the multiple little q-Laguerre polynomials. This shows that $T_{b,c}q_{\vec{n}}(x;\frac{b}{q}\vec{a}|q)$ satisfies the multiple orthogonality conditions with respect to the r continuous dual q-Hahn weights $w(\theta;a_1,b,c|q),\dots,w(\theta;a_r,b,c|q)$. The r-fold sum is obtained by applying $T_{b,c}$ to the r-fold sum (4.82) for the multiple little q-Laguerre polynomials. □

Observe that for $c=0$ one has the multiple Al-Salam–Chihara polynomials given in Theorem 4.16.

4.2.3.5 Multiple Askey–Wilson polynomials

Finally we will obtain the multiple Askey–Wilson polynomials by extending Theorem 4.8. We choose the r weights $(w_1,\dots,w_r)$ on $[-1,1]$ by taking the Askey–Wilson weights $w(\theta;a_j,b,c,d|q)$ of (4.54) with r different parameters $\vec{a}=(a_1,\dots,a_r)$, keeping b,c,d fixed. Of course we could have taken r different parameters $(b_1,\dots,b_r)$, keeping a,c,d fixed or r different c or d parameters, but since the Askey–Wilson weight is symmetric in (a,b,c,d) this would not give

anything new. Again we let $a_j = q^{\alpha_j}$ and assume that $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$ to ensure that the multiple orthogonality conditions

$$\int_{-1}^{1} p_{\vec{n}}(x;\vec{a},b,c,d|q)x^{\ell}w(\theta;a_j,b,c,d|q)\,\frac{dx}{\sqrt{1-x^2}} = 0, \qquad 0 \leq \ell \leq n_j - 1,$$

for $1 \leq j \leq r$, give $|\vec{n}|$ equations that determine the $p_{\vec{n}}(x;\vec{a},b,c|q)$ uniquely (up to a multiplicative factor).

Theorem 4.18. *The multiple Askey–Wilson polynomials $p_{\vec{n}}(x;\vec{a},b,c,d|q)$ for the weights $w(\theta;a_1,b,c,d|q), w(\theta;a_2,b,c,d|q),\ldots,w(\theta;a_r,b,c,d|q)$, with $w(\theta;a,b,c,d|q)$ given in (4.54), are given by $p_{\vec{n}}(x;\vec{a},b,c,d|q) = T_{b,c,d}q_{\vec{n}}(x;\frac{b}{q}\vec{a},\frac{cd}{q}|q)$, where $T_{b,c,d}$ is the linear transformation given in Theorem 4.9 and $q_{\vec{n}}(x;\frac{b}{q}\vec{a},\frac{cd}{q}|q)$ are the multiple little q-Jacobi polynomials given in Section 4.2.3.2. An explicit expression is given by*

$$p_{\vec{n}}(x;\vec{a},b,c,d|q) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\prod_{j=1}^{r}\frac{(q^{-n_j};q)_{k_j}(a_jbcdq^{\sum_{i=1}^{j}n_i-1};q)_{\sum_{i=j}^{r}k_i}}{(q;q)_{k_j}(a_jb;q)_{\sum_{i=j}^{r}k_i}}$$
$$\times\prod_{j=1}^{r-1}\frac{(a_jbq^{n_j};q)_{k_{j+1}+\cdots+k_r}}{(a_jbcdq^{\sum_{i=1}^{j}n_i-1};q)_{k_{j+1}+\cdots+k_r}}\frac{q^{|\vec{k}|}}{q^{\sum_{j=1}^{r}n_j\sum_{i=j+1}^{r}k_i}}\frac{(be^{i\theta},be^{-i\theta};q)_{|\vec{k}|}}{(bc;q)_{|\vec{k}|}(bd;q)_{|\vec{k}|}}.$$

Proof. Proposition 4.10 gives

$$\langle T_{a_j,c,d}x^{\ell}, T_{b,c,d}q_{\vec{n}}(x;\frac{b}{q}\vec{a},\frac{cd}{q}|q)\rangle_{\mathrm{cont}(a_j,b,c,d)} = \langle x^{\ell}, q_{\vec{n}}(x;\frac{b}{q}\vec{a},\frac{cd}{q}|q)\rangle_{\mathrm{dis}(a_jb,cd)},$$

which is

$$\frac{(a_jc,a_jd,bc,bd,cd;q)_{\infty}}{2\pi}\int_{-1}^{1}\frac{(a_je^{i\theta},a_je^{-i\theta};q)_{\ell}}{(a_jc,a_jd;q)_k}T_{b,c,d}q_{\vec{n}}(x;\frac{b}{q}\vec{a},\frac{cd}{q}|q)w(\theta;a_j,b,c,d|q)\,\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{(q;q)_{\infty}}\sum_{k=0}^{\infty}q^{k\ell}q_{\vec{n}}(q^k;\frac{b}{q}\vec{a},\frac{cd}{q}|q)(a_jb)^k\frac{(cd;q)_k}{(q;q)_k}$$

and the latter sum is 0 whenever $0 \leq \ell \leq n_j - 1$ because of the multiple orthogonality condition (4.87) for the multiple little q-Jacobi polynomials. This shows that $T_{b,c,d} q_{\vec{n}}(x; \frac{b}{q}\vec{a}, \frac{cd}{q}|q)$ satisfies the multiple orthogonality conditions with respect to the r continuous dual Askey–Wilson weights $w(\theta; a_1, b, c, d|q), \ldots, w(\theta; a_r, b, c, d|q)$. The r-fold sum is obtained by applying $T_{b,c,d}$ to the r-fold sum (4.89) for the multiple little q-Jacobi polynomials, where we first replace b by cd/q and then a_j by ba_j/q. □

Observe that for $d = 0$ one has the multiple continuous dual q-Hahn polynomials given in Theorem 4.17, and for $c = d = 0$ the multiple Al-Salam–Chihara polynomials given in Theorem 4.16.

Chapter 5

q-Nonuniform difference integral inequalities

In this this chapter, as in [21], we will first establish q-nonuniform versions of some integral inequalities of classical mathematical analysis such as the integral inequalities of Hölder, Cauchy-Schwarz and Minkowski. It can be seen that the techniques used in classical analysis remain valid here. Then we establish q-nonuniform versions of some other integral inequalities based on the linear q-nonuniform difference equations of first order and the corresponding Lagrange resolution method: the inequalities of Grönwall, Bernoulli, and finally prove the Lyapunov inequality for the solutions of the q-nonuniform Sturm-Liouville equation. The difference versions of the same integral inequalities were established in [20].

5.1 q-Nonuniform Hölder and Cauchy-Shwartz inequalities

Theorem 5.1. *(q-**nonuniform Hölder inequality**): Let $a, b \in [1, \infty[\cap\mathbb{T}$. For all real valued functions f, g, defined and q-nonuniform*

integrable on $[a,b]$*, we have:*

$$\int_a^b |f(x(z))g(x(z))|d_qx(z) \leqslant \Big(\int_a^b |f(x(z))|^\alpha d_qx(z)\Big)^{\frac{1}{\alpha}} \Big(\int_a^b |g(x(z))|^\beta d_qx(z)\Big)^{\frac{1}{\beta}} \tag{5.1}$$

with $\alpha > 1$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. Let's first show that for $A, B \in [1, \infty[$ we have

$$A^{\frac{1}{\alpha}} B^{\frac{1}{\beta}} \leqslant \frac{A}{\alpha} + \frac{B}{\beta}. \tag{5.2}$$

Indeed, since $\frac{1}{\alpha}+\frac{1}{\beta} = 1$, $A, B \in [1, \infty[$, $\frac{A^\alpha}{\alpha}+\frac{B^\beta}{\beta}$ runs through the segment $[A^\alpha, B^\beta]$ while $\frac{\log A^\alpha}{\alpha}+\frac{\log B^\beta}{\beta}$ runs throuth the segment linking the points $(A^\alpha, \log A^\alpha)$ and $(A^\beta, \log A^\beta)$. By concavity of the logarithm function, we conclude that $\log\left(\frac{A^\alpha}{\alpha}+\frac{B^\beta}{\beta}\right) \geqslant \frac{\log A^\alpha}{\alpha}+\frac{\log B^\beta}{\beta} = \log(AB)$. By applying the exponential to the two members of this inequality, we obtain (5.2).

Let us now take

$$A(x(z)) = \frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha d_qx(z)} \quad \textit{and} \quad B(x(z)) = \frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta d_qx(z)},$$

considering that

$$\Big(\int_a^b |f(x(z))|^\alpha d_qx(z)\Big)\Big(\int_a^b |g(x(z))|^\beta d_qx(z)\Big) \neq 0 \tag{5.3}$$

(otherwise, clearly $f \equiv 0$ or $g \equiv 0$ and (5.1) becomes equality). Due to the Property 1.1 and its Corollary on the positivity of the integral,

by substituting $A(x(z))$ and $B(x(z))$ in (5.2), and integrating on $[a,b]$, we have

$$\int_a^b \frac{|f(x(z))|}{\Big(\int_a^b |f(x(z))|^\alpha d_q x(z)\Big)^{\frac{1}{\alpha}}} \frac{|g(x(z))|}{\Big(\int_a^b |g(x(z))|^\beta d_q x(z)\Big)^{\frac{1}{\beta}}} d_q x(z)$$

$$= \int_a^b A^{\frac{1}{\alpha}} B^{\frac{1}{\beta}} d_q x(z) \leqslant \int_a^b \Big\{\frac{A}{\alpha} + \frac{B}{\beta}\Big\} d_q x(z)$$

$$= \int_a^b \Big\{\frac{1}{\alpha} \frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha d_q x(z)} + \frac{1}{\beta} \frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta d_q x(z)}\Big\} d_q x(z)$$

$$= \frac{1}{\alpha} \int_a^b \Big\{\frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha d_q x(z)}\Big\} d_q x(z) + \frac{1}{\beta} \int_a^b \Big\{\frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta d_q x(z)}\Big\} d_q x(z)$$

$$= \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

which gives us directly the q-nonuniform Hölder inequality

$$\int_a^b |f(x(z))g(x(z))| d_q x(z) \leqslant$$

$$\Big(\int_a^b |f(x(z))|^\alpha d_q x(z)\Big)^{\frac{1}{\alpha}} \Big(\int_a^b |g(x(z))|^\beta d_q x(z)\Big)^{\frac{1}{\beta}}. \tag{5.4}$$

□

If we take $\alpha = \beta = 2$ in the q-nonuniform Hölder inequality (5.1), we have the q-nonuniform Cauchy-Shwartz inequality.

Corollary 5.2. *(q-**nonuniform Cauchy-Schwartz inequality**): Let $a, b \in [1, \infty[\cap\mathbb{T}$. For all real valued functions f, g, defined and q-nonuniform integrable on $[a, b]$, we have:*

$$\int_a^b |f(x(z))g(x(z))|d_q x(z) \leqslant \sqrt{\Big(\int_a^b |f(x(z))|^2 d_q x(z)\Big)\Big(\int_a^b |g(x(z))|^2 d_q x(z)\Big)}. \tag{5.5}$$

5.2 q-Nonuniform Minkowski inequality

We can now use the q-nonuniform Hölder inequality to deduce the q-nonuniform Minkowski inequality:

Theorem 5.3. *(q-**nonuniform Minkowski inequality**): Let $a, b \in [1, \infty[\cap\mathbb{T}$. For all real valued functions f, g, defined and q-nonuniform integrable on $[a, b]$, we have*

$$\Big(\int_a^b |(f+g)(x(z))|^\alpha d_q x(z)\Big)^{\frac{1}{\alpha}} \leqslant$$
$$\Big(\int_a^b |f(x(z))|^\alpha d_q x(z)\Big)^{\frac{1}{\alpha}} + \Big(\int_a^b |g(x(z))|^\alpha d_q x(z)\Big)^{\frac{1}{\alpha}} \tag{5.6}$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ where $\alpha > 1$ and $\beta > 1$.

Proof. We have

$$\begin{aligned}\int_a^b |(f+g)(x(z))|^\alpha d_q x(z) &= \int_a^b |(f+g)(x(z))|^{\alpha-1}|(f+g)(x(z))|d_q x(z)\\ &\leqslant \int_a^b |(f+g)(x(z))|^{\alpha-1}|f(x(z))|d_q x(z)\\ &\quad + \int_a^b |(f+g)(x(z))|^{\alpha-1}|g(x(z))|d_q x(z).\end{aligned} \tag{5.7}$$

Using the q-nonuniform Hölder inequality with $\beta = \frac{\alpha}{\alpha-1}$, we will obtain

$$\begin{aligned}\int_a^b |(f+g)(x(z))|^\alpha d_q x(z) &\leqslant \Big\{\int_a^b |f(x(z))|^\alpha d_q x(z)\Big\}^{\frac{1}{\alpha}}\Big\{\int_a^b |(f+g)(x(z))|^{(\alpha-1)\beta}|d_q x(z)\Big\}^{\frac{1}{\beta}}\\ &\quad +\Big\{\int_a^b |g(x(z))|^\alpha d_q x(z)\Big\}^{\frac{1}{\alpha}}\Big\{\int_a^b |(f+g)(x(z))|^{(\alpha-1)\beta}|d_q x(z)\Big\}^{\frac{1}{\beta}}\\ &= \Big[\Big\{\int_a^b |f(x(z))|^\alpha d_q x(z)\Big\}^{\frac{1}{\alpha}} + \Big\{\int_a^b |g(x(z))|^\alpha d_q x(z)\Big\}^{\frac{1}{\alpha}}\Big]\Big\{\int_a^b |(f+g)(x(z))|^\alpha |d_q x(z)\Big\}^{\frac{1}{\beta}}.\end{aligned} \tag{5.8}$$

Dividing both sides of this inequality by $\Big\{\int_a^b |(f+g)(x(z))|^\alpha |d_q x(z)\Big\}^{\frac{1}{\beta}}$, we have

$$\Big(\int_a^b |(f+g)(x(z))|^\alpha d_q x(z) \Big)^{1-\frac{1}{\beta}} \leqslant$$

$$\Big\{ \int_a^b |f(x(z))|^\alpha d_q x(z) \Big\}^{\frac{1}{\alpha}} + \Big\{ \int_a^b |g(x(z))|^\alpha d_q x(z) \Big\}^{\frac{1}{\alpha}}. \tag{5.9}$$

As $1 - \frac{1}{\beta} = \frac{1}{\alpha}$, this gives us the q-nonuniform Minkowski inequality.

□

5.3 q-Nonuniform Grönwall inequality

Let's first introduce the following inequalities based on the Lagrange method for the linear q-nonuniform difference non-homogeneous equations.

Lemma 5.4. *Let y, f be real valued functions defined and q-nonuniform integrable on $[c,d]$, $\forall c,d \in [1,\infty[\cap\mathbb{T}$. Let $a(x(z))$ such that $p(x(z)) = 1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(z - z^{-1})a(x(z)) > 0$. That means $a(x(z)) \geqslant 0$.*

Suppose that $y_0(x(z))$ is a solution of

$$\mathcal{D}y_0(x(z)) = a(x(z))y_0(x(zq^{-\frac{1}{2}})) \tag{5.10}$$

such that $y_0(x(z_0)) = 1$.

If

$$\mathcal{D}y(x(z)) \leqslant a(x(z))y(x(zq^{-\frac{1}{2}})) + f(x(z)) \qquad \forall x(z) \in [1,\infty[, \tag{5.11}$$

then

$$y(x(z)) \leqslant y_0(x(z))y(x(z_0)) + y_0(x(z)) \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{\frac{1}{2}}))f(x(t))d_q x(t). \tag{5.12}$$

Proof. Let $y_0(x(z))$ be the solution of the homogeneous equation

$$\mathcal{D}y_0(x(z)) = a(x(z))y_0(x(zq^{-\frac{1}{2}})) \tag{5.13}$$

such that $y_0(x(z_0)) = 1$. By looking for the function $y(x(z))$ satisfying (5.11) by the Lagrange method,

$$y(x(z)) = c(x(z))y_0(x(z)) \tag{5.14}$$

where $c(x(z))$ is indeterminate, we replace (5.14) in (5.11) and we have:

$$\mathcal{D}\Big[c(x(z))y_0(x(z))\Big] \leqslant a(x(z))\Big[c(x(zq^{\frac{1}{2}}))y_0(x(zq^{-\frac{1}{2}}))\Big] + f(x(z)) \tag{5.15}$$

or

$$\begin{aligned} y_0(x(zq^{\frac{1}{2}}))\mathcal{D}c(x(z)) + c(x(zq^{-\frac{1}{2}}))\mathcal{D}y_0(x(z)) \leqslant \\ a(x(z))c(x(z))y_0(x(zq^{\frac{1}{2}})) + f(x(z)). \end{aligned} \tag{5.16}$$

Using (5.13), we have

$$\begin{aligned} y_0(x(zq^{\frac{1}{2}}))\mathcal{D}c(x(z)) + c(x(zq^{-\frac{1}{2}}))a(x(z))y_0(x(zq^{-\frac{1}{2}})) \\ \leqslant a(x(z))c(x(zq^{-\frac{1}{2}}))y_0(x(zq^{-\frac{1}{2}})) + f(x(z)). \end{aligned} \tag{5.17}$$

Simplifying this gives

$$y_0(x(zq^{\frac{1}{2}}))\mathcal{D}c(x(z)) \leqslant f(x(z)) \tag{5.18}$$

or

$$\mathcal{D}c(x(z)) \leqslant y_0^{-1}(x(zq^{\frac{1}{2}}))f(x(z)), \tag{5.19}$$

since $a(x(z)) \in \mathbb{R}^+$ implies $y_0(x(z)) > 0$. By integrating the two members of the equality from $x(z_0)$ to $x(z)$, we will have:

$$c(x(z)) - c(x(z_0)) \leqslant \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{\frac{1}{2}}))f(x(t))d_qx(t). \tag{5.20}$$

But, from (5.14) follows $c(x(z_0)) = y(x(z_0))$, given that $y_0(x(z_0)) = 1$. Hence

$$c(x(z)) \leqslant y(x(z_0)) + \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{\frac{1}{2}}))f(x(t))d_qx(t). \tag{5.21}$$

Placing (5.21) in (5.14), we get

$$y(x(z)) = c(x(z))y_0(x(z)) \leqslant$$
$$y_0(x(z))\Big[y(x(z_0)) + \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{\frac{1}{2}}))f(x(t))d_qx(t)\Big], \tag{5.22}$$

which gives the desired result

$$y(x(z)) \leqslant y_0(x(z))y(x(z_0)) + y_0(x(z)) \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{\frac{1}{2}}))f(x(t))d_qx(t). \tag{5.23}$$

□

Taking account of theorem 2.2, and the fact that by the definition $E_{a,q^{-\frac{1}{2}}}(x(z_0); x(z_0)) = E_{a,q^{\frac{1}{2}}}(x(z_0); x(z_0)) = 1$, we obtain

Corollary 5.5. *If y, f and a are functions satisfying conditions of Lemma 5.4, then*

$$y(x(z)) \leqslant E_{a;q^{-\frac{1}{2}}}(x(z_0), x(z))y(x(z_0))$$
$$+E_{a;q^{-\frac{1}{2}}}(x(z_0), x(z)) \int_{x(z_0)}^{x(z)} E_{-a;q^{\frac{1}{2}}}(x(z_0), x(tq^{\frac{1}{2}}))f(x(t))d_q x(t). \quad (5.24)$$

Lemma 5.6. *Let y, f be real valued functions defined and q-non unifom integrable on $[c, d]$, $\forall c, d \in [1, \infty[\cap\mathbb{T}$. Let $a(x(z))$ such that*

$$p(z) = 1 - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(z - z^{-1})a(x(z)) > 0.$$

This means that $a(x(z)) \leqslant 0$.
Suppose that $y_0(x(z))$ is a solution of

$$\mathcal{D}y_0(x(z)) = a(x(z))y_0(x(zq^{\frac{1}{2}}))$$

such that $y_0(x(z_0)) = 1$.
In that case, if

$$\mathcal{D}y(x(z)) \leqslant a(x(z))y((x(zq^{\frac{1}{2}})) + f(x(z)), \qquad \forall x(z) \in [1, \infty[, \quad (5.25)$$

then, for all $x(z) \in [1, \infty[$ we have

$$y(x(z)) \leqslant y_0(x(z))y(x(z_0)) + y_0(x(z)) \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{-\frac{1}{2}}))f(x(t))d_q x(t). \quad (5.26)$$

Proof. Let $y_0(x(z))$ be a solution of the homogeneous equation

$$\mathcal{D}y_0(x(z)) = a(x(z))y_0(x(zq^{\frac{1}{2}})), \qquad y_0(x(z_0)) = 1. \quad (5.27)$$

Looking for the function $y(x(z))$ satisfying (5.25) by the Lagrange method,

$$y(x(z)) = c(x(z))y_0(x(z)) \tag{5.28}$$

where $c(x(z))$ is indeterminate, we replace the function $y(x(z))$ in (5.25) and we have:

$$\mathcal{D}\Big[c(x(z))y_0(x(z))\Big] \leqslant a(x(z))c(x(zq^{\frac{1}{2}}))y_0(x(zq^{\frac{1}{2}})) + f(x(z))$$

or

$$\begin{gathered} c(x(zq^{\frac{1}{2}}))\mathcal{D}y_0(x(z)) + y_0(x(zq^{-\frac{1}{2}}))\mathcal{D}c(x(z)) \leqslant \\ a(x(z))c(x(zq^{-\frac{1}{2}}))y_0(x(zq^{-\frac{1}{2}})) + f(x(z)). \end{gathered}$$

Using (5.27), we have

$$\begin{gathered} c(x(zq^{\frac{1}{2}}))a(x(z))y_0(x(zq^{\frac{1}{2}})) + y_0(x(zq^{-\frac{1}{2}}))\mathcal{D}c(x(z)) \leqslant \\ a(x(z))c(x(zq^{\frac{1}{2}}))y_0(x(zq^{\frac{1}{2}})) + f(x(z)). \end{gathered} \tag{5.29}$$

Simplifying this gives

$$y_0(x(zq^{-\frac{1}{2}}))\mathcal{D}c(x(z)) \leqslant f(x(z)) \tag{5.30}$$

or

$$\mathcal{D}c(x(z)) \leqslant y_0^{-1}(x(zq^{-\frac{1}{2}}))f(x(z)) \tag{5.31}$$

since $a(x(z)) \leqslant 0$ implies $y_0(x(z)) > 0$. By integrating the two members of the equality above from $x(z_0)$ to $x(z)$, we will have:

$$c(x(z)) - c(x(z_0)) \leqslant \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{-\frac{1}{2}}))f(x(t))d_qx(t). \tag{5.32}$$

But, from (5.28) follows $c(x(z_0)) = y(x(z_0))$, given that $y_0(x(z_0)) = 1$. This gives

$$c(x(z)) \leqslant y(x(z_0)) + \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{-\frac{1}{2}}))f(x(t))d_q x(t). \tag{5.33}$$

Then, we obtain

$$y(x(z)) = c(x(z))y_0(x(z)) \leqslant$$
$$y_0(x(z))\Big[y(x(z_0)) + \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{-\frac{1}{2}}))f(x(t))d_q x(t)\Big], \tag{5.34}$$

which gives the desired result:

$$y(x(z)) \leqslant y_0(x(z))y(x(z_0)) + y_0(x(z)) \int_{x(z_0)}^{x(z)} y_0^{-1}(x(tq^{-\frac{1}{2}}))f(x(t))d_q x(t). \tag{5.35}$$

□

As well as for the corrollary 5.5, we deduce

Corollary 5.7. *If y, f and a satisfy the conditions of the Lemma 5.6, then*

$$y(x(z)) \leqslant E_{a;q^{\frac{1}{2}}}(x(z_0), x(z))y(x(z_0)) +$$
$$E_{a;q^{\frac{1}{2}}}(x(z_0), x(z)) \int_{x(z_0)}^{x(z)} E_{-a;q^{-\frac{1}{2}}}(x(z_0), x(tq^{-\frac{1}{2}}))f(x(t))d_q x(t). \tag{5.36}$$

Theorem 5.8. *(**q-nonuniform Grönwall inequality**): Let y, f be real valued and q-non unifom integrable functions on $[c, d]$, $\forall c, d \in$*

$[1, \infty[\cap\mathbb{T}$, *and* $a \geqslant 0$. *If*

$$y(x(z)) \leqslant f(x(z)) + \int_{x(z_0)}^{x(z)} y(x(tq^{-\frac{1}{2}}))a(x(t))d_q x(t) \tag{5.37}$$

then

$$y(x(z)) \leqslant f(x(z))+$$
$$E_{a,q^{-\frac{1}{2}}}(x(z_0), x(z)) \int_{x(z_0)}^{x(z)} a(x(t))f(x(tq^{-\frac{1}{2}}))E_{-a,q^{\frac{1}{2}}}(x(z_0); x(tq^{\frac{1}{2}}))d_q x(t). \tag{5.38}$$

Proof. Let's define

$$\nu(x(z)) = \int_{x(z_0)}^{x(z)} y(x(tq^{-\frac{1}{2}}))a(x(t))d_q x(t). \tag{5.39}$$

Then $\nu(x(z_0)) = 0$ and $\mathcal{D}\nu = y(x(zq^{-\frac{1}{2}}))a(x(z))$.

Hence the hypothesis (5.37) gives

$$y(x(z)) \leqslant f(x(z)) + \nu(x(z)) \tag{5.40}$$

and

$$\mathcal{D}\nu(x(z)) = y(x(zq^{-\frac{1}{2}}))a(x(z)) \leqslant \left[f(x(zq^{-\frac{1}{2}})) + \nu(x(zq^{-\frac{1}{2}}))\right]a(x(z)), \tag{5.41}$$

or

$$\mathcal{D}\nu(x(z)) \leqslant f(x(zq^{-\frac{1}{2}}))a(x(z)) + \nu(x(zq^{-\frac{1}{2}}))a(x(z)). \tag{5.42}$$

From Lemma 5.4, the inequality (5.42) gives

$$\nu(x(z)) \leqslant \nu(x(z_0))E_{a,q^{-\frac{1}{2}}}(x(z_0);x(z))+$$
$$E_{a,q^{-\frac{1}{2}}}(x(z_0);x(z))\int_{x(z_0)}^{x(z)} a(x(t))f(x(tq^{-\frac{1}{2}}))E_{-a,q^{\frac{1}{2}}}(x(z_0);x(tq^{\frac{1}{2}}))d_qx(t), \tag{5.43}$$

and the inequality (5.40) implies that (with $\nu(x(z_0)) = 0$)

$$y(x(z)) \leqslant f(x(z))+$$
$$E_{a,q^{-\frac{1}{2}}}(x(z_0);x(z))\int_{x(z_0)}^{x(z)} a(x(z))f(x(tq^{-\frac{1}{2}}))E_{-a,q^{\frac{1}{2}}}(x(z_0);x(tq^{\frac{1}{2}}))d_qx(t), \tag{5.44}$$

which is the q-nonuniform Grönwall inequality. □

As a direct consequence, we get the following results

Corollary 5.9. *Let y, f be real valued functions defined and q-nonuniform integrable on $[c,d]$, $\forall c,d \in [1,\infty[\cap\mathbb{T}$ and $a(x(z))$ such that $1+(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(z-z^{-1})a(x(z)) > 0$. That means $a(x(z)) \geqslant 0$. Then*

$$y(x(z)) \leqslant \int_{x(z_0)}^{x(z)} y(x(tq^{-\frac{1}{2}}))a(x(t))d_qx(t). \tag{5.45}$$

for all $x(z)$, implies that $y(x(z)) \leqslant 0$.

Proof. This is due to the Theorem 5.8 with $f(x(z)) \equiv 0$ □

Corollary 5.10. *Let $a \geqslant 0$ and $\alpha \in \mathbb{R}$. If*

$$y(x(z)) \leqslant \alpha + \int_{x_0}^{x(z)} y(x(tq^{-\frac{1}{2}}))a(x(t))d_q x(t) \tag{5.46}$$

for all $x_0 = x(z_0) > 0$, then $y(x(z)) \leqslant \alpha E_{a,q^{-\frac{1}{2}}}(x_0, x(z))$.

Proof. By the q-nonuniform Grönwall integral inequality (5.38), if we take $f(x(z)) = \alpha$, then

$$\begin{aligned}
y(x(z)) &\leqslant \alpha + E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \int_{x_0}^{x(z)} \alpha a(x(t)) E_{-a,q^{\frac{1}{2}}}(x_0, x(tq^{\frac{1}{2}}))d_q x(t) \\
&= \alpha\Big(1 - E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \int_{x_0}^{x(z)} \mathcal{D}E_{-a,q^{\frac{1}{2}}}(x_0, x(z))d_q x(t)\Big) \\
&= \alpha\Big(1 - E_{a,q^{-\frac{1}{2}}}(x_0, x(z))\Big(E_{-a,q^{\frac{1}{2}}}(x_0, x(z)) - E_{-a,q^{\frac{1}{2}}}(x_0, x_0)\Big)\Big) \\
&= \alpha - \alpha E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) E_{-a;q^{\frac{1}{2}}}(x_0; x(z)) + \alpha E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \\
&= \alpha E_{a,q^{-\frac{1}{2}}}(x_0, x(z)).
\end{aligned}$$

□

5.4 q-Nonuniform Bernoulli inequality

Theorem 5.11. *(q-**nonuniform Bernoulli inequality**): Let $\alpha \in \mathbb{R}$. Then $\forall x(z)$, $x_0 = x(z_0) \in [1, \infty[$ with $x(zq^{-\frac{1}{2}}) > x_0$ we have*

$$E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \geqslant 1 + \alpha(x(z) - x_0). \tag{5.47}$$

Proof. Let's take $y(x(z)) = \alpha(x(z) - x_0), \quad x(zq^{-\frac{1}{2}}) > x_0, \quad \forall z$. Then $\mathcal{D}y(x(z)) = \alpha$ and we have:

$$\alpha y(x(zq^{-\frac{1}{2}})) + \alpha = \alpha^2(x(zq^{-\frac{1}{2}}) - x_0) + \alpha \geqslant \alpha = \mathcal{D}y(x(z)),$$

which implies that $\mathcal{D}y(x(z)) \leqslant \alpha y(x(zq^{-\frac{1}{2}})) + \alpha$.

On the other hand, by Lemma 5.4, we get (with $y(x_0) = 0$)

$$\begin{aligned}
y(x(z)) &\leqslant y(x_0)E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \\
&\quad + E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \int_{x_0}^{x(z)} aE_{-a,q^{\frac{1}{2}}}(x_0, x(tq^{\frac{1}{2}}))d_q x(t) \\
&= E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \int_{x_0}^{x(z)} (-)\mathcal{D}E_{-a,q^{\frac{1}{2}}}(x_0, x(t))d_q x(t) \\
&= -E_{a,q^{-\frac{1}{2}}}(x_0, x(z))\Big(E_{-a,q^{\frac{1}{2}}}(x_0, x(z)) - 1\Big) \\
&= -1 + E_{a,q^{-\frac{1}{2}}}(x_0, x(z)).
\end{aligned}$$

That is why $E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \geqslant 1 + y(x(z)) = 1 + \alpha(x(z) - x_0)$. □

5.5 q-Nonuniform Lyapunov inequality

Let us now turn to q-nonuniform Lyapunov inequality regarding the derivative and integral introduced. For that, consider the following q-nonuniform Sturm-Liouville equation,

$$\mathcal{D}\mathcal{D}^{+}u(x(z)) + f(x(z))u(x(zq^{\frac{1}{2}})) = 0 \tag{5.48}$$

where D^+ is defined by $\mathcal{D}^+ f(x(z)) = \frac{f(x(zq))-f(x(z))}{x(zq)-x(z)}$.

Let's define the function F by

$$F(y) = \int_a^b \Big[\Big(\mathcal{D}y(x(z))\Big)^2 - f(x(z))\Big(y(x(zq^{\frac{1}{2}}))\Big)^2\Big] d_q x(z). \tag{5.49}$$

Lemma 5.12. *Let u be a non-trivial solution of the q-nonuniform Sturm-Liouville equation (5.48). Then for all y belonging to the domain of F, the following equality remains verified:*

$$F(y) - F(u) - F(y-u) = 2(y-u)(b)\mathcal{D}^+u(b) - 2(y-u)(a)\mathcal{D}^+u(a). \tag{5.50}$$

Proof. We have

$$\begin{aligned}
F(y) - F(u) - F(y-u) = \int_a^b & \Big\{\Big(\mathcal{D}y(x(z))\Big)^2 - f(x(z))\Big(y(x(q^{\frac{1}{2}}z))\Big)^2 \\
& - \Big(\mathcal{D}u(x(z))\Big)^2 - f(x(z))\Big(y(x(q^{\frac{1}{2}}z))\Big) - \Big(\mathcal{D}(y-u)(x(z))\Big)^2 \\
& - f(x(z))\Big((y-u)(x(q^{\frac{1}{2}}z))\Big)^2\Big\} d_q x(z) \\
= 2\int_a^b & \Big\{ - \Big(\mathcal{D}u(x(z))\Big)^2 + f(x(z))\Big(u(x(zq^{\frac{1}{2}}))\Big)^2 + \mathcal{D}y(x(z))\mathcal{D}u(x(z)) \\
& - f(x(z))y(x(zq^{\frac{1}{2}}))u(x(zq^{\frac{1}{2}}))\Big\} d_q x(z)
\end{aligned}$$

$$= 2\int_a^b \Big\{ -\Big(\mathcal{D}u(x(z))\Big)^2 - u(x(zq^{\frac{1}{2}}))\mathcal{D}\mathcal{D}^+u(x(z)) + \mathcal{D}y(x(z))\mathcal{D}u(x(z))$$
$$+ y(x(zq^{\frac{1}{2}}))\mathcal{D}\mathcal{D}^+u(x(z))\Big\}d_qx(z)$$
$$= 2\int_a^b \Big\{\mathcal{D}y(x(z))\mathcal{D}u(x(z)) + y(x(zq^{\frac{1}{2}}))\mathcal{D}\mathcal{D}^+u(x(z))$$
$$- \Big[\Big(\mathcal{D}u(x(z))\Big)^2 + u(x(zq^{\frac{1}{2}}))\mathcal{D}\mathcal{D}^+u(x(z))\Big]\Big\}d_qx(z),$$

(using $f(x(z))u(x(zq^{\frac{1}{2}})) = -\mathcal{D}\mathcal{D}^+u(x(z))$),

or

$$F(y) - F(u) - F(y-u)$$
$$= 2\int_a^b \Big\{\mathcal{D}\Big[y(x(z))\mathcal{D}^+u(x(z))\Big] - \mathcal{D}\Big[u(x(z))\mathcal{D}^+u(x(z))\Big]\Big\}d_qx(z)$$
$$= 2\int_a^b \mathcal{D}\Big\{\Big(y(x(z)) - u(x(z))\Big)\mathcal{D}^+u(x(z))\Big\}d_qx(z)$$
$$= 2\Big(y(b) - u(b)\Big)\mathcal{D}^+u(b) - 2\Big(y(a) - u(a)\Big)\mathcal{D}^+u(a).$$

□

Lemma 5.13. *Let y be in the domain of F, then $\forall a, b \in [1, \infty[\cap\mathbb{T}$ and $c, d \in [a, b] \cap \mathbb{T}$ such that $a \leqslant c \leqslant d \leqslant b$, we have*

$$\int_c^d \Big(\mathcal{D}y(x(z))\Big)^2 d_qx(z) \geqslant \frac{\Big(y(d) - y(c)\Big)^2}{d-c}. \tag{5.51}$$

Proof. Let's take

$$u(x(z)) = \frac{y(d) - y(c)}{d-c}x(z) + \frac{dy(c) - cy(d)}{d-c}.$$

Then $\mathcal{D}^+u(x(z)) = \frac{y(d)-y(c)}{d-c}$ and $\mathcal{D}\mathcal{D}^+u(x(z)) = 0$. Which proves that $u(x(z))$ is a solution of (5.48) with $f(x(z)) = 0$, $\forall x(z) \in [1,\infty[\cap\mathbb{T}$ and

$$F(y) = \int_a^b \Big(\mathcal{D}y(x(z))\Big)^2 d_q x(z),$$

$\forall y(x(z))$ from the domain of F. From Lemma 5.12, we get

$$F(y)-F(u)-F(y-u) = 2(y-u)(b)\mathcal{D}^+u(b)-2(y-u)(a)\mathcal{D}^+u(a) = 0.$$

Consequently

$$F(y) = F(u) + F(y-u) \geqslant F(u).$$

This leads us to the following result

$$\begin{aligned}\int_c^d \Big(\mathcal{D}y(x(z))\Big)^2 d_q x(z) &\geqslant \int_c^d \Big(\mathcal{D}u(x(z))\Big)^2 d_q x(z)\\ &= \int_c^d \Big(\frac{y(d)-y(c)}{d-c}\Big)^2 d_q x(z)\\ &= \frac{\Big(y(d)-y(c)\Big)^2}{d-c}.\end{aligned}$$

□

Theorem 5.14. *(q-**nonuniform Lyapunov inequality**): Let f be a real valued function defined and q-nonuniform differentiable on $[a,b]$, $\forall a,b \in [1,\infty[\cap\mathbb{T}$, $a<b$ and let $u(x(z))$ be a non-trivial solution of the equation (5.48) with $u(a) = u(b) = 0$, then*

$$\int_a^b f(x(z))d_q x(z) = -\int_a^b \frac{\mathcal{D}\mathcal{D}^+u(x(z))}{u(x(zq^{\frac{1}{2}}))}d_q x(z) \geqslant \frac{4}{b-a}. \qquad (5.52)$$

Proof. From Lemma 5.12 with $y = 0$ and $u(a) = u(b) = 0$, we have

$$F(u) = \int_a^b \Big[\Big(\mathcal{D}u(x(z))\Big)^2 - f(x(z))\Big(u(x(zq^{\frac{1}{2}}))\Big)^2\Big] d_q x(z) = 0. \quad (5.53)$$

Let $M = max\{u^2(x(zq^{\frac{1}{2}})); x(q^{\frac{1}{2}}z) \in [a,b] \cap \mathbb{T}\}$ and $c \in [a,b] \cap \mathbb{T}$ such that $u^2(c) = M$. Then

$$M = u^2(c) \geqslant u^2(x(zq^{\frac{1}{2}})) > 0,$$

and

$$\begin{aligned}
M \int_a^b f(x(z)) d_q x(z) &\geqslant \int_a^b f(x(z)) u^2(x(zq^{\frac{1}{2}})) d_q x(z) \\
&= \int_a^b \Big(\mathcal{D}u(x(z))\Big)^2 d_q x(z) \\
&= \int_a^c \Big(\mathcal{D}u(x(z))\Big)^2 d_q x(z) + \int_c^b \Big(\mathcal{D}u(x(z))\Big)^2 d_q x(z) \\
&\geqslant \frac{\Big(u(c) - u(a)\Big)^2}{c-a} + \frac{\Big(u(b) - u(c)\Big)^2}{b-c} \\
&= M\Big[\frac{1}{c-a} + \frac{1}{b-c}\Big] \quad (u(a) = u(b) = 0 \ \ and \ \ u^2(c) = M) \\
&= M\Big[\frac{(a+b-2c)^2}{(c-a)(b-c)(b-a)} + \frac{4}{b-a}\Big] \\
&\geqslant M\frac{4}{b-a},
\end{aligned}$$

which implies that $\int_a^b f(x(z)) d_q x(z) \geqslant M\frac{4}{b-a}$.
The q-nonuniform Lyapunov inequality is thus proved. □

Chapter 6

q-Nonuniform Laplace transform

For a given function $f(x(z))$, we define its *q-nonuniform Laplace transform* or *Laplace transform on the q-nonuniform lattice* $x(z) = (z + z^{-1})/2$, $z = q^s$, $0 < q < 1$, $-\infty \leq s \leq +\infty$, as the function [19]

$$\begin{aligned} F(p) &= \mathcal{L}_{q^{\frac{1}{2}}}\{f(x(z))\} \\ &\overset{\text{def}}{=} \int_{x(z_0)=x_0}^{x(\infty)=\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; zq^{-\frac{1}{2}}) f(x(z)) dx(z);\ p = s + i\sigma \in \mathbb{C}. \end{aligned}$$

We denote $f(x(z)) \rightleftharpoons_{q^{\frac{1}{2}}} F(p)$, and we say that $f(x(z))$ is the *original* of $F(p)$ while $F(p)$ is the *image* of $f(x(z))$ by the Laplace transform on q-nonuniform lattices.

6.1 Properties of the q-nonuniform Laplace transform

6.1.1 Linearity.

By the linearity of the integral, it becomes clear that

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\alpha f(x(z)) + \beta g(x(z))\} = \alpha \mathcal{L}_{q^{\frac{1}{2}}}\{f(x(z))\} + \beta \mathcal{L}_{q^{\frac{1}{2}}}\{g(x(z))\}.$$

6.1.2 Transform of derivatives.

We have

$$\begin{aligned}\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}f(x(z))\} &= \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p;z_0;zq^{-\frac{1}{2}})\mathcal{D}f(x(z))dx(z)\\ &= \Big[E_{q^{-\frac{1}{2}}}(-p;z_0;z)f(x(z))\Big]_{z_0}^{\infty}\\ &\quad -\int_{x_0}^{\infty} f(x(zq^{\frac{1}{2}}))\mathcal{D}E_{q^{-\frac{1}{2}}}(-p;z_0;z)dx(z)\\ &= -f(x(z_0))+p\int_{x_0}^{\infty} f(x(zq^{\frac{1}{2}}))E_{q^{-\frac{1}{2}}}(-p;z_0;zq^{-\frac{1}{2}})dx(z)\\ &= p\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\frac{1}{2}}))\}-f(x(z_0)), \qquad (6.1)\end{aligned}$$

where we used the q-nonuniform integration by parts:

$$\begin{aligned}&\int_{x_0}^{x(z)} g(x(zq^{-\frac{1}{2}}))\mathcal{D}f(x(z))dx(z)\\ &= [g(x(z))f(x(z))]_{x_0}^{x(z)}-\int_{x_0}^{x(z)} f(x(zq^{\frac{1}{2}}))\mathcal{D}g(x(z))dx(z), \qquad (6.2)\end{aligned}$$

and (2.78). Thus

1)

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}f(x(z))\}=p\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\frac{1}{2}}))\}-f(x(z_0)), \qquad (6.3)$$

and

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}f(x(zq^{\alpha}))\}=p\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\alpha+\frac{1}{2}}))\}-f(x(q^{\alpha}z_0)). \qquad (6.4)$$

Hence

$$\mathcal{D}f(x(zq^{-\frac{1}{2}}))\rightleftharpoons_{q^{\frac{1}{2}}} pF(p)-f(x(z_0q^{-\frac{1}{2}})), \qquad (6.5)$$

and

$$\mathcal{D}f(x(zq^{\frac{1}{2}})) \rightleftharpoons_{q^{\frac{1}{2}}} p\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq))\} - f(x(z_0q^{\frac{1}{2}})). \tag{6.6}$$

Using (6.3), one easily obtains

2)

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^2 f(x(z))\} = \mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}(\mathcal{D}f(x(z)))\}$$
$$= p^2\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq))\} - pf(x(q^{\frac{1}{2}}z_0)) - \mathcal{D}f(x(z_0)). \tag{6.7}$$

Hence

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^2 f(x(zq^{\frac{1}{2}}))\} =$$
$$p^2\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\frac{3}{2}}))\} - pf(x(qz_0)) - \mathcal{D}f(x(q^{\frac{1}{2}}z_0)). \tag{6.8}$$

3)

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^3 f(x(z))\} = \mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}(\mathcal{D}^2 f(x(z)))\} = p^3\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\frac{3}{2}}))\}$$
$$-[p^2 f(x(qz_0)) + p\mathcal{D}f(x(q^{\frac{1}{2}}z_0)) + \mathcal{D}^2 f(x(z_0))]. \tag{6.9}$$

...

...

...

n)

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^n f(x(z))\} = p^n\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(zq^{\frac{n}{2}}))\}$$
$$-\left[p^{n-1}f(x(q^{\frac{n-1}{2}}z_0)) + p^{n-2}\mathcal{D}f(x(q^{\frac{n-2}{2}}z_0)) + \cdots + \mathcal{D}^{n-1}f(x(z_0))\right], \tag{6.10}$$

or equivalently

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^n f(x(q^{-\frac{n}{2}}z))\} = p^n \mathcal{L}_{q^{\frac{1}{2}}}\{f(x(z))\}$$
$$- \left[p^{n-1} f(x(q^{-\frac{1}{2}}z_0)) + p^{n-2}\mathcal{D}f(x(q^{-1}z_0)) + \cdots + \mathcal{D}^{n-1} f(x(q^{-\frac{n}{2}}z_0))\right]. \quad (6.11)$$

For example, for $n = 2$, we have

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^2 f(x(zq^{-1}))\} =$$
$$p^2 \mathcal{L}_{q^{\frac{1}{2}}}\{f(x(z))\} - [pf(x(q^{-\frac{1}{2}}z_0)) + \mathcal{D}f(x(q^{-1}z_0))]. \quad (6.12)$$

6.1.3 Transform of integrals.

In the equation

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}f(x(z))\} = p\mathcal{L}_{q^{\frac{1}{2}}}\{f(x(q^{\frac{1}{2}}z))\} - f(x(z_0)),$$

set

$$f(x(z)) = \int_{x_0}^{x(z)} g(x(z))dx(z). \quad (6.13)$$

We have

$$\mathcal{L}_{q^{\frac{1}{2}}}\{g(x(z))\} = p\mathcal{L}_{q^{\frac{1}{2}}}\{\int_{x_0}^{x(zq^{\frac{1}{2}})} g(x(z))dx(z)\}. \quad (6.14)$$

Hence

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\int_{x_0}^{x(zq^{\frac{1}{2}})} g(x(z))dx(z)\} = \frac{1}{p}\mathcal{L}_{q^{\frac{1}{2}}}\{g(x(z))\}. \quad (6.15)$$

6.2 q-Nonuniform Laplace transform for some elementary functions

1. $f(x(z)) = 1$. We have

$$\begin{aligned}
\mathcal{L}_{q^{\frac{1}{2}}}\{1\} &= \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)dx(z) \\
&= -\frac{1}{p}\int_{x_0}^{\infty} \mathcal{D}E_{q^{-\frac{1}{2}}}(-p; z_0; z)dx(z) \\
&= -\frac{1}{p}E_{q^{-\frac{1}{2}}}(-p; z_0; z)\mid_{z_0}^{\infty} \\
&= \frac{1}{p}. \qquad (6.16)
\end{aligned}$$

2. $f(x(z)) = x(zq^{\frac{1}{2}})$. We have

$$\begin{aligned}
&\mathcal{L}_{q^{\frac{1}{2}}}\{x(zq^{\frac{1}{2}})\} \\
&= \int_{x_0}^{\infty} x(zq^{\frac{1}{2}})E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)dx(z) \\
&= -\frac{1}{p}\int_{x_0}^{\infty} x(zq^{\frac{1}{2}})\mathcal{D}E_{q^{-\frac{1}{2}}}(-p; z_0; z)dx(z) \\
&= -\frac{1}{p}\left[x(z)E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} + \frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)dx(z) \\
&= \frac{x(z_0)}{p} - \frac{1}{p^2}\int_{x_0}^{\infty} \mathcal{D}E_{q^{-\frac{1}{2}}}(-p; z_0; z)dx(z) \\
&= \frac{x(z_0)}{p} - \frac{1}{p^2}\left[E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} \\
&= \frac{x(z_0)}{p} + \frac{1}{p^2}. \qquad (6.17)
\end{aligned}$$

3. $f(x(z)) = E_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)$. We calculate

$$\begin{aligned}
I = & \ \mathcal{L}_{q^{\frac{1}{2}}}\{E_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} \\
= & \ \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z) E_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) dx(z) \\
= & \ -\frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) \mathcal{D}E_{q^{-\frac{1}{2}}}(-p; z_0; z) dx(z) \\
= & \ -\frac{1}{p}\left[E_{q^{-\frac{1}{2}}}(a; z_0; z) E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} \\
& + \frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z) \mathcal{D}E_{q^{\frac{1}{2}}}(a; x_0; z) dx(z) \\
= & \ \frac{1}{p} + \frac{a}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z) E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) dx(z). \\
= & \ \frac{1}{p} + \frac{a}{p} I.
\end{aligned}$$

Hence

$$I = \frac{1}{p-a}. \tag{6.18}$$

4. $f(x(z)) = \cos_{q^{-\frac{1}{2}}}(ai; z_0; q^{\frac{1}{2}}z) = \frac{E_{q^{-\frac{1}{2}}}(ai; z_0; q^{\frac{1}{2}}z) + E_{q^{-\frac{1}{2}}}(-ai; z_0; q^{\frac{1}{2}}z)}{2}$. Using the preceding case, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\cos_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} = \frac{1}{2}\left(\frac{1}{p-ai} + \frac{1}{p+ai}\right) = \frac{p}{p^2+a^2}. \tag{6.19}$$

5. $f(x(z)) = \sin_{q^{-\frac{1}{2}}}(ai; z_0; q^{\frac{1}{2}}z) = \frac{E_{q^{-\frac{1}{2}}}(ai; z_0; q^{\frac{1}{2}}z) - E_{q^{-\frac{1}{2}}}(-ai; z_0; q^{\frac{1}{2}}z)}{2i}$. Similarly, we get

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\sin_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} = \frac{1}{2i}\left(\frac{1}{p-ai} - \frac{1}{p+ai}\right) = \frac{a}{p^2+a^2}. \tag{6.20}$$

6. $f(x(z)) = \cosh_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) = \frac{E_{q^{-\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)+E_{q^{-\frac{1}{2}}}(-a;z_0;q^{\frac{1}{2}}z)}{2}$. Here also, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\cosh_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} = \frac{1}{2}\left(\frac{1}{p-a} + \frac{1}{p+a}\right) = \frac{p}{p^2-a^2}. \quad (6.21)$$

7. $f(x(z)) = \sinh_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) = \frac{E_{q^{-\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)-E_{q^{-\frac{1}{2}}}(-a;z_0;q^{\frac{1}{2}}z)}{2}$. Similarly,

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\sinh_{q^{-\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} = \frac{1}{2}\left(\frac{1}{p-a} - \frac{1}{p+a}\right) = \frac{a}{p^2-a^2}. \quad (6.22)$$

8. $f(x(z)) = E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)$. We calculate

$$\begin{aligned}
I = \;& \mathcal{L}_{q^{\frac{1}{2}}}\{E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} \\
= \;& \int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)dx(z) \\
= \;& -\frac{1}{p}\int_{x_0}^{\infty} E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\mathcal{D}E_{q^{-\frac{1}{2}}}(-p; z_0; z)dx(z) \\
= \;& -\frac{1}{p}\left[E_{q^{\frac{1}{2}}}(a; z_0; z)E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} \\
& +\frac{1}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)\mathcal{D}E_{q^{\frac{1}{2}}}(a; x_0; z)dx(z) \\
= \;& -\frac{1}{p}\left[E_{q^{\frac{1}{2}}}(a; z_0; z)E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} \\
& +\frac{a}{p}\int_{x_0}^{\infty} E_{q^{-\frac{1}{2}}}(-p; z_0; q^{-\frac{1}{2}}z)E_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)dx(z). \\
= \;& -\frac{1}{p}\left[E_{q^{\frac{1}{2}}}(a; z_0; z)E_{q^{-\frac{1}{2}}}(-p; z_0; z)\right]_{z_0}^{\infty} + \frac{a}{p}I.
\end{aligned} \quad (6.23)$$

Hence

$$\begin{aligned}(1-\frac{a}{p})I &= -\frac{1}{p}\left[E_{q^{-\frac{1}{2}}}(-p;z_0;z)E_{q^{\frac{1}{2}}}(a;x_0;z)\right]_{z_0}^{\infty}\\ &= \frac{1}{p}-\frac{1}{p}E_{q^{-\frac{1}{2}}}(-p;z_0;\infty)E_{q^{\frac{1}{2}}}(a;z_0;\infty).\end{aligned}$$

Consequently

$$I=\frac{1}{p-a}-\frac{1}{p-a}E_{q^{-\frac{1}{2}}}(-p;z_0;\infty)E_{q^{\frac{1}{2}}}(a;z_0;\infty)=\frac{1-K_{-p,a}}{p-a} \quad (6.24)$$

where $K_{-p,a}=E_{q^{-\frac{1}{2}}}(-p;z_0;\infty)E_{q^{\frac{1}{2}}}(a;z_0;\infty)$ (see (2.88)).

9. $f(x(z))=\cos_{q^{\frac{1}{2}}}(ai;z_0;q^{\frac{1}{2}}z)=\frac{E_{q^{\frac{1}{2}}}(ai;z_0;q^{\frac{1}{2}}z)+E_{q^{\frac{1}{2}}}(-ai;z_0;q^{\frac{1}{2}}z)}{2}$. Using the preceding case, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\cos_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)\}=\frac{1}{2}\left(\frac{1-K_{-p,ai}}{p-ai}+\frac{1-K_{-p,-ai}}{p+ai}\right). \quad (6.25)$$

10. $f(x(z))=\sin_{q^{\frac{1}{2}}}(ai;z_0;q^{\frac{1}{2}}z)=\frac{E_{q^{\frac{1}{2}}}(ai;z_0;q^{\frac{1}{2}}z)-E_{q^{\frac{1}{2}}}(-ai;z_0;q^{\frac{1}{2}}z)}{2i}$. Similarly, we get

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\sin_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)\}=\frac{1}{2i}\left(\frac{1-K_{-p,ai}}{p-ai}-\frac{1-K_{-p,-ai}}{p+ai}\right). \quad (6.26)$$

11. $f(x(z))=\cosh_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)=\frac{E_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)+E_{q^{\frac{1}{2}}}(-a;z_0;q^{\frac{1}{2}}z)}{2}$. Here also, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\cosh_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z)\}=\frac{1}{2}\left(\frac{1-K_{-p,a}}{p-a}+\frac{1-K_{-p,-a}}{p+a}\right). \quad (6.27)$$

12. $f(x(z)) = \sinh_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z) = \frac{E_{q^{\frac{1}{2}}}(a;z_0;q^{\frac{1}{2}}z) - E_{q^{\frac{1}{2}}}(-a;z_0;q^{\frac{1}{2}}z)}{2}$. Similarly,

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\sinh_{q^{\frac{1}{2}}}(a; z_0; q^{\frac{1}{2}}z)\} = \frac{1}{2}\left(\frac{1 - K_{-p,a}}{p-a} - \frac{1 - K_{-p,-a}}{p+a}\right). \quad (6.28)$$

6.3 Inverse of the q-nonuniform Laplace transform

For searching the original function from its image, we have to use, as in other cases of Laplace transforms, the q-nonuniform Laplace transform properties of the subsection 6.1 and the q-nonuniform Laplace transform of elementary functions of the subsection 6.2.

6.4 Applications of q-nonuniform Laplace transform on q-nonuniform difference equations

In the same way that the Laplace transfrom, Z-transform and q-Laplace transfrom are applied respectively in differential, difference and q-difference equations [15], the q-nonuniform Laplace transform is expected to be applied in q-nonuniform difference equations [14]. For question of simplicity, we suppose that the order of the equation is $n = 2$.

So, consider the second order q-nonuniform difference equation

$$a_0\mathcal{D}^2 y(x(q^{-1}z)) + a_1\mathcal{D}y(x(q^{-\frac{1}{2}}z)) + a_2 y(x(z)) = g(x(z)), \quad (6.29)$$

with the initial conditions

$$y(x(q^{-\frac{1}{2}}z_0)) = y_0;\ \mathcal{D}y(x(q^{-1}z_0)) = y_1. \quad (6.30)$$

Applying the q-nonuniform Laplace transform as in the subsection 6.1, we obtain

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}^2 y(x(q^{-1}z))\} =$$
$$p^2\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} - py(x(q^{-\frac{1}{2}}z_0)) - \mathcal{D}y(x(q^{-1}z_0)), \tag{6.31}$$

$$\mathcal{L}_{q^{\frac{1}{2}}}\{\mathcal{D}y(x(q^{-\frac{1}{2}}z))\} = p\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} - y(x(q^{-\frac{1}{2}}z_0)). \tag{6.32}$$

Applying the q-nonuniform Laplace transform on both sides of the equation (6.29) and using (6.30), (6.31) and (6.32), we get

$$\begin{aligned} &a_0p^2\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} - a_0y_0p - a_0y_1 + a_1p\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} \\ &\quad -a_1y_0 + a_2\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} = G(p) \\ \Leftrightarrow\ &a_0p^2\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} + a_1p\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} + a_2\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} \\ &\quad = G(p) + a_0y_0p + a_0y_1 + a_1y_0. \end{aligned}$$

Setting $\mathcal{L}_{q^{\frac{1}{2}}}\{y(x(z))\} = Y(p)$, we obtain

$$Y(p)(a_0p^2 + a_1p + a_2) = G(p) + a_0y_0p + a_0y_1 + a_1y_0.$$

Consequently, $Y(p) = \frac{G(p)+a_0y_0p+a_0y_1+a_1y_0}{a_0p^2+a_1p+a_2}$ and $y(x(z)) = \mathcal{L}^{-1}_{q^{\frac{1}{2}}}\{Y(p)\}$.

Conclusion and outlooks

The first major concern of this thesis (the first three chapters) deals with the introduction of the basic concepts of q-nonuniform difference calculus such as the differentiation and integrationn on q-nonuniform lattices and the use of those concepts to solve q-nonuniform difference equations of first order, systems of linear q-nonuniform difference equations and q-nonuniform difference equations of higher order. We introduce also the basic concepts of q-nonuniform difference control theory and its applications.

In Chapter 4, we have given a number of new families of multiple orthogonal polynomials starting from the discrete multiple little q-Jacobi polynomials and working our way up to the multiples Askey–Wilson polynomials.

In Chapter 5, q-nonuniform difference versions of the integral inequalities of Hölder, Cauchy-Shwartz, Minkowski, Grönwall, Bernoulli and Lyapunov were proved.

Finaly, in Chapter 6, the concepts of q-nonuniform difference version of the integral Laplace transform and some of its applications were discussed.

An interesting outlook consists in using the transformation in Section 4.2 of (multiple) little q-Jacobi polynomials, to transform modified (multiple) little q-Jacobi polynomials (which have a non classical weight) [18] in modified (multiple) Askey-Wilson polynomials (admitting a non classical weight).

Bibliography

[1] A. I. Aptekarev. Multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 99(1-2):423–447, 1998.

[2] A. I. Aptekarev, A. Branquinho, and W. Van Assche. Multiple orthogonal polynomials for classical weights. *Transactions of the American Mathematical Society*, 355(10):3887–3914, 2003.

[3] A. I. Aptekarev, M. Derevyagin, H. Miki, W. Van Assche, et al. Multidimensional Toda lattices: continuous and discrete time. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 12:054, 2016.

[4] A. I. Aptekarev, M. Derevyagin, and W. Van Assche. On 2D discrete Schrödinger operators associated with multiple orthogonal polynomials. *Journal of Physics A: Mathematical and Theoretical*, 48(6):065201, 2015.

[5] A. I. Aptekarev, G. Lagomasino, and A. Martínez-Finkelshtein. On Nikishin systems with discrete components and weak asymptotics of multiple orthogonal polynomials. *Russian Mathematical Surveys*, 72(3(435)):389–449, 2017.

[6] J. Arvesú. On some properties of q-Hahn multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 233(6):1462–1469, 2010.

[7] J. Arvesú, J. Coussement, and W. Van Assche. Some discrete multiple orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 153(1-2):19–45, 2003.

[8] J. Arvesú and A. M. Ramírez-Aberasturis. On the q-Charlier multiple orthogonal polynomials. *arXiv preprint arXiv:1411.2000*, 2014.

[9] R. Askey and J. Wilson. Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials. *Mem. Am. Math. Soc.*, 54:1–55, 1985.

[10] M. N. Atakishiyev and S. K. Suslov. Difference hypergeometric functions. In *Progress in Approximation Theory*, pages 1–35. Springer, 1992.

[11] G. Bangerezako. The factorization method for the Askey–Wilson polynomials. *Journal of Computational and Applied Mathematics*, 107(2):219–232, 1999.

[12] G. Bangerezako. Variational q-calculus. *Journal of Mathematical Analysis and Applications*, 289(2):650–665, 2004.

[13] G. Bangerezako. Variational calculus on q-nonuniform lattices. *Journal of Mathematical Analysis and Applications*, 306(1):161–179, 2005.

[14] G. Bangerezako. An introduction to q-nonuniform difference equations. *Preprint, Triestre, Miramare*, 2011.

[15] G. Bangerezako. q-Difference linear control systems. *Journal of Difference Equations and Applications*, 17(9):1229–1249, 2011.

[16] G. Bangerezako. q-Nonuniform difference linear control systems. *Journal of Applied Analysis*, 20(2):133–143, 2014.

[17] G. Bangerezako. An introduction to q-difference equations. *UCL, IMPA; Séminaire de Mathématiques*, Rapport n. 354, November 2008.

[18] G. Bangerezako and M. N. Hounkonnou. The transformation of polynomial eigenfunctions of linear second-order q-difference operators: a special case of q-Jacobi polynomials. Contemporary problems in mathematical physics (Cotonou, 2001), 427-439. *Wold Sci. Publishing, River Edge, NJ*, 2002.

[19] G. Bangerezako and J. P. Ntahomvukiye. q-Nonuniform Laplace transform. *Submitted*, 2019.

[20] G. Bangerezako and J. P. Nuwacu. Some difference integral inequalities. In *Mathematical Structures and Applications, (T Diagana, B Toni, eds.)*, pages 137–152. Springer, 2018.

[21] G. Bangerezako and J. P. Nuwacu. q-Nonuniform difference calculus and classical integral inequalities. *Journal of Inequalities and Applications*, 28, 2019.

[22] R. J. Barro and X. Sala-i Martin. Economic growth. *MIT Press, Cambridge*, 1999.

[23] B. Beckermann, J. Coussement, and W. Van Assche. Multiple Wilson and Jacobi–Piñeiro polynomials. *Journal of Approximation Theory*, 132(2):155–181, 2005.

[24] P. M. Bleher and A. B. J. Kuijlaars. Large n limit of Gaussian random matrices with external source, Part I. *Communications in Mathematical Physics*, 252(1-3):43–76, 2004.

[25] P. M. Bleher and A. B. J. Kuijlaars. Random matrices with external source and multiple orthogonal polynomials. *International Mathematics Research Notices*, 2004(3):109–129, 2004.

[26] J. A. Cadzow. Discrete calculus of variations. *International Journal of Control*, 11(3):393–407, 1970.

[27] S. Elaydi. *An introdution to difference equations.* Springer-Verlag, New York, 1995.

[28] L. Euler. Introductio in analysin infinitorum. *Lousane*, 1, 1078.

[29] R. V. Gamkrelidze, L. S. Pontryaguin, and V. G. Boltjanskij. *The mathematical theory of optimal processes.* Macmillan Company, 1964.

[30] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 96. Cambridge University Press, 2004.

[31] M. E. H. Ismail. Classical and quantum orthogonal polynomials in one variable. *Encyclopedia of Mathematics and its Applications*, 98, 2005.

[32] M. E. H. Ismail and E. Koelink. Spectral properties of operators using tridiagonalization. *Analysis and Applications*, 10(03):327–343, 2012.

[33] H. F. Jackson. q-Difference equations. *Am. J. Math.*, 32:305–314, 1910.

[34] V. Kac and P. Cheung. *Quantum calculus.* Springer Science & Business Media, 2002.

[35] E. G. Kalnins and W. Miller Jr. Symmetry techniques for q-series: Askey-Wilson polynomials. *The Rocky Mountain Journal of Mathematics*, pages 223–230, 1989.

[36] A. P. Kartaschev and Rojestvinie. Differential equations and variational calculus. *Nauka, Moscou*, 1986.

[37] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[38] R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. 1998.

[39] E. Koelink and J. V. Stokman. The Askey-Wilson function transform. *International Mathematics Research Notices*, 2001(22):1203–1227, 2001.

[40] T. H. Koornwinder. Special orthogonal polynomial systems mapped onto each other by the Fourier-Jacobi transform. In *Polynômes Orthogonaux et Applications, (C Brezinski et al., eds.), Lecture Notes in Mathematics* ***1171***, pages 174–183. Springer, 1985.

[41] A. B. J. Kuijlaars. Multiple orthogonal polynomial ensembles. *Recent trends in orthogonal polynomials and approximation theory, Contemp. Math*, 507:155–176, 2010.

[42] A. B. J. Kuijlaars. Multiple orthogonal polynomials in random matrix theory. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 1417–1432. World Scientific, 2010.

[43] A. P. Magnus. Associated Askey-Wilson polynomials as Lageurre-Hahn orthogonal polynomials. *Springer Lectures Notes in Math., Springer Berlin*, 1329:261–278, 1988.

[44] A. P. Magnus. Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points. *Journal of Computational and Applied Mathematics*, 65(1-3):253–265, 1995.

[45] A. Martínez-Finkelshtein and W. Van Assche. What is . . . a multiple orthogonal polynomial? *arXiv preprint arXiv:1707.09511*, (9):1029–1031, 2017.

[46] H. Miki, S. Tsujimoto, L. Vinet, and A. Zhedanov. An algebraic model for the multiple Meixner polynomials of the first kind. *Journal of Physics A: Mathematical and Theoretical*, 45(32):325205, 2012.

[47] H. Miki, L. Vinet, and A. Zhedanov. Non-hermitian oscillator Hamiltonians and multiple Charlier polynomials. *Physics Letters A*, 376(2):65–69, 2011.

[48] A. B. Milonowska and D. F. Torres. The quantum variational calculus. *Journal of Optimization Theory and Applications*, 147(3):419–442, 2010.

[49] F. Ndayiragije and W. Van Assche. Multiple Meixner polynomials and non-Hermitian oscillator Hamiltonians. *Journal of Physics A: Mathematical and Theoretical*, 46(50):505201, 2013.

[50] A. F. Nikiforov, V. B. Uvarov, and S. K. Suslov. *Classical orthogonal polynomials of a discrete variable*. Springer, 1991.

[51] E. M. Nikishin and V. N. Sorokin. *Rational approximations and orthogonality*. Amer Mathematical Society, 1991.

[52] J. P. Nuwacu and W. Van Assche. Multiple Askey-Wilson polynomials and related basic hypergeometric multiple orthogonal polynomials. *arXiv preprint arXiv:1904.01252.*, 2019.

[53] K. Postelmans and W. Van Assche. Multiple little q-Jacobi polynomials. *Journal of Computational and Applied Mathematics*, 178(1-2):361–375, 2005.

[54] K. Postelmans and W. Van Assche. Irrationality of $\zeta_q(1)$ and $\zeta_q(2)$. *Journal of Number Theory*, 126(1):119–154, 2007.

[55] C. Smet and W. Van Assche. Irrationality proof of a q-extension of $\zeta(2)$ using little q-Jacobi polynomials. 138(2):165–178, 2009.

[56] C. Smet and W. Van Assche. Mellin transforms for multiple Jacobi-Piñeiro polynomials and a q-analogue. *Journal of Approximation Theory*, 162(4):782–806, 2010.

[57] J. V. Stokman. An expansion formula for the Askey–Wilson function. *Journal of Approximation Theory*, 114(2):308–342, 2002.

[58] W. Van Assche and E. Coussement. Some classical multiple orthogonal polynomials. *Numerical analysis 2000, Vol. V, Quadrature and Orthogonal Polynomials, J. Comput. Appl. Math.*, 127(1–2):317–347, 2001.