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Multiple Hermite Polynomials andtheir Applications in RandomMatrices with External Source

HATUNGIMANA, Samuel

UB

UNIVERSITÉ DU BURUNDI

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Département de Mathématiques

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Samuel HATUNGIMANA

MÉMOIRE

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Sous la direction de: **Professeur Walter Van Assche (directeur)**
Professeur François NDAYIRAGIJE (co-directeur)

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Dedication

To

my Parents:

Augustin MINANI

and

Valérie SIBOMANA

my Wife

Sandrine NDAYISHIMIYE

And my Daughter

Schilo-Samuella AKEZA

my Brothers and Sisters

All my Family

my Schoolmates

All my Acquaintances

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List of acronyms and abbreviations

GUE: Gaussian Unitarly Ensemble

KU: Katolieke Universiteite Leuven

RH: Riemann-Hilbert

Abstract

Multiple orthogonal polynomials are polynomials of one variable which are defined by orthogonality relations with respect to r different measures $\mu_1, \mu_2, \ldots, \mu_r$ where $r \geq 1$ is a positive integer. Multiple Hermite polyniomials play an important role in analysing random matrices with external source and also in the analysis of non intersecting Brownian motions leaving from one point and arriving at r distinct points. In this work, we focus on random matrices with external source and we show that multiple Hermite polynomials belong to a problem of such a matrix ensemble and we give some applications of multiple Hermite polynomials. The main contributions of this work are contained in Theorem 3.0.3 of the Chapter 3 where we corrected the Theorem 2.4. of the paper of Lee, we also give two other proofs for the generating function and in Chapter 6 where we derived the asymptotic behaviors of multiple Hermite polynomials in region II and III inside of the disk $D(z_1, r)$ of the complex plane.

Key words: multiple Hermite polynomials, generating function, random matrices with external source, Riemann-Hilbert problem, asymptotic behavior.

Résumé

Les polynômes multiples d'Hermite sont des polynômes avec une seule variable définies par les relations d'orthogonalité par rapport à r mesures positives différentes. Ces polynômes jouent un rôle important dans l'analyse des matrices aléatoires avec source extérieure mais aussi dans l'analyse des mouvements non-Browniens partant d'un point vers r points différents.Dans ce travail, nous nous focalisons sur les matrices aléatoires avec source extérieure et montrons que les polynômes multiples d'Hermite appartiennent dans un problème avec cet ensemble de matrices et nous donnons quelques applications. Les principales contributions de ce travail sont contenues dans le Théorème 3.0.3 du Chapitre 3 où nous donnons la forme correcte du Théorème 2.4 de l'article de Lee, nous donnons aussi deux autres démonstrations de la fonction génératrice et dans le Chapitre 6 nous dérivons les comportements asymptotiques des polynômes multiples d'Hermite dans les régions II et III à l'intérieur du disque $D(z_1, r)$ du plan complexe.

Mots clés: polynômes multiples d'Hermite, fonction génératrice, matrices aléatoires avec source extérieure, problème de Riemann-Hilbert, comportement asymptotique.

Condensé en Français

Polynômes orthogonaux classiques

Considérons l'équation différentielle

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

où σ est un polynôme de degré au plus 2, τ un polynôme de degré 1 et λ une constante. Les polynômes orthogonaux sont des polynômes satisfaisant

$$\int_a^b p_n(x)p_m(x)d\mu(x) = \delta_{mn}, \quad m, n \geq 0.$$

Les polynômes orthogonaux satisfont toujours une relation de récurrence à trois terme

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

Les polynômes orthogonaux moniques $P_n(x) = \frac{p_n(x)}{\gamma_n}$ satisfont aussi une relation de récurrence à trois terme de la forme

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

La formule de Rodrigues est donnée par

$$w(x)p_n(x) = c_n \frac{d^n}{dx^n}(w(x)\sigma^n(x)).$$

Résumé des polynômes orthogonaux classiques: Hermite, Laguerre et Jacobi

Polynômes orthogonaux d'Hermite

Les polynômes d'Hermite $H_n(x)$ de degré n sont des solutions polynomiales de l'équation différentielle

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

	Hermite $H_n(x)$	Laguerre $L_n^{\alpha}(x)$	Jacobi $P_n^{(\alpha,\beta)}(x)$
interval	$(-\infty,\infty)$	$[0,\infty)$	$[-1,1]$
$w(x)$	e^{-x^2}	$x^{\alpha}e^{-x}$	$(1-x)^{\alpha}(1+x)^{\beta}$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-x(\alpha+\beta+2)-\alpha+\beta$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
c_n	$(-1)^n$	$\frac{1}{n!}$	$\frac{(-1)^n}{2^n n!}$
Remark	$-$	$\alpha>-1$	$\alpha,\beta>-1$

Ces polynômes satisfont la relation d'orthogonalité

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{mn}.$$

La relation de récurrence à trois terme est données par

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Sa formule de Rodrigues est

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}).$$

L'expression explicite des polynômes d'Hermite est donnée par

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}$$

et la fonction génératrice est

$$G(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^n = e^{2xt-t^2}$$

que l'on peut généraliser par

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x)}{n!}t^n = e^{2xt-t^2}H_k(x-t)$$

Polynômes orthogonaux multiples

Les polynômes multiples de type I pour un multi-indice $\vec{n}$ consistent en un vecteur $(A_{\vec{n},1},...,A_{\vec{n},r})$ de r polynômes, où $A_{\vec{n},j}$ est de degré au plus n_j-1 pour $1\le j\le r$, pour lesquels

$$\int x^k \sum_{j=1}^{r} A_{\vec{n},j}(x)d\mu_j(x) = 0, \quad 0 \le k \le |\vec{n}|-2$$

avec une normalisation

$$\int x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x) d\mu_j(x) = 1.$$

Nous notons par $Q_{\vec{n}}$ la fonction de type I définie par

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x)\omega_j(x).$$

Les polynômes multiples de type II pour un multi-indice $\vec{n}$ sont des polynômes moniques $P_{\vec{n}}$ de degré $\leq \mid \vec{n} \mid$ for pour lesquels

$$\int P_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad 0 \leq k \leq n_j - 1, \quad \text{for } 1 \leq j \leq r.$$

Les polynômes $P_{\vec{n}}$ et $Q_{\vec{m}}$ sont biorthogonaux et les relations de biorthogonalité sont données par

$$\int P_{\vec{n}}(x) Q_{\vec{m}}(x) d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \leq \vec{n} \\ 0, & \text{if } |\vec{n}| \leq |\vec{m}| - 2 \\ 1, & \text{if } |\vec{n}| = |\vec{m}| - 1 \end{cases}$$

Les relations de recurrence pour les polynômes multiples de type I sont données par

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x), \quad j = 1, 2, ..., r.$$

Les relations de recurrence pour les polynômes multiples de type II sont données par

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x), \quad j = 1, 2, ..., r.$$

Polynômes multiples d'Hermite

Les polynômes multiples d'Hermite jouent un rôle important dans l'analyse des matrices aléatoires avec source extérieure. Les polynômes multiples de type I pour un multi-indice $\vec{n}$ consistent en un vecteur $(A_{\vec{n},1}, ..., A_{\vec{n},r})$ of r polynomials, where $A_{\vec{n},j}$ de degré au plus $n_j - 1$ for $1 \leq j \leq r$, pour lesquels

$$\int_{-\infty}^{+\infty} x^k \sum_{j=1}^{r} A_{\vec{n},j}(x) e^{-x^2 + c_j x} dx = 0, \quad 0 \leq k \leq |\vec{n}| - 2$$

avec la normalisation

$$\int_{-\infty}^{+\infty} x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x) e^{-x^2 + c_j x} dx = 1.$$

Les polynômes multiples de type II pour un multi-indice $\vec{n}$ sont des polynômes moniques $H_{\vec{n}}$ de degré $\leq |\vec{n}|$ pour lesquels

$$\int_{-\infty}^{+\infty} H_{\vec{n}}(x)x^k e^{-x^2+c_jx}dx = 0, \quad 0 \leq k \leq n_j - 1,\ 1 \leq j \leq r.$$

La formule de Rodrigues pour ces polynômes est donnée par

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}}(\prod_{j=1}^{r} e^{-c_jx}\frac{d^{n_j}}{dx^{n_j}}e^{c_jx})e^{-x^2}$$

et l'expression explicite s'écrit

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r}c_1^{n_1-k_1}\ldots c_r^{n_r-k_r}(-1)^{|\vec{k}|}H_{|\vec{k}|}(x).$$

On a les opérateurs de création de la forme

$$\frac{d}{dx}(e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x)) = -2e^{-x^2+c_jx}H_{\vec{n}}(x), \quad 1 \leq j \leq r$$

et les opérateurs d'annihilation de la forme

$$\frac{d}{dx}(H_{\vec{n}}(x)) = \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x).$$

Combinant les deux opérateurs, on trouve une équation différentielle d'ordre $r+1$

$$\Big(\prod_{j=1}^{r} D_j\Big)DH_{\vec{n}}(x) = -2\Big(\sum_{j=1}^{r} n_j\prod_{i\neq j} D_i\Big)H_{\vec{n}}(x).$$

En utilisant la formule de Rodrigues, on peut établir la fonction $g_r(t)$ donnée par

$$g_r(t) = \int_{\mathbb{R}} e^{tx}H_{\vec{n}}(x)e^{-x^2}dx = 2^{-|\vec{n}|}\sqrt{\pi}(t-c_1)^{n_1}...(t-c_r)^{n_r}e^{t^2/4}.$$

Cette fonction nous permet de calculer les coefficients de récurrence et nous trouvons les relations de récurrence de la forme

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e_j}}(x) + \frac{c_j}{2}H_{\vec{n}}(x) + \frac{1}{2}\sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e_j}}(x), \quad 1 \leq j \leq r.$$

Les polynômes multiples d'Hermite de type I ont pour représentation intégrale

$$A_{\vec{n},j}(x)e^{-x^2+c_j} = \frac{1}{\sqrt{\pi}2\pi i}\oint_{\Gamma_j} e^{-(t-x)^2}\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}dt$$

où Γ_j est un contour fermé encerclant $\frac{c_j}{2}$ dans la direction positive mais n'encerclant pas tout autre $\frac{c_l}{2}, l \neq j$.
. La fonction $Q_{\vec{n}}$ de type I a pour représentation intégrale

$$Q_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}2\pi i} \oint_\Gamma e^{-(t-x)^2} \prod_{l=1}^{r} (t - \frac{c_l}{2})^{-n_l} dt$$

où Γ est un contour fermé encerclant tous les $\frac{c_l}{2}, (1 \leq l \leq r)$ dans la direction positive.
Les polynômes multiples d'Hermite de type II ont pour représentation intégrale

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi} i} \int_{-i\infty}^{i\infty} e^{(s-x)^2} \prod_{l=1}^{r} (s - \frac{c_l}{2})^{n_l} ds.$$

Fonction génératrice des polynômes multiples d'Hermite

La fonction génératrice des polynômes multiples d'Hermite a été trouvée par Lee [2] pour $t_i \to t_i/2$ et sa preuve utilisait la formule de Rodrigues et la formule intégrale de Cauchy. Cette fonction génératrice est ainsi définie par

$$G(x; \mathbf{t}) = \exp\Big(x \sum_{j=1}^{r} t_j - \frac{1}{4} (\sum_{j=1}^{r} t_j)^2 - \frac{1}{2} \sum_{j=1}^{r} c_j t_j\Big).$$

Dans ce travail, nous donnons deux autres méthodes de démonstration, l'une basée sur la fonction $g_r(t)$ déjà définie et l'autre sur l'expression explicite.
Le Théorème 2.4 dans [2] est erroné, voici la version correcte.

Theorem 0.0.1. Pour les polynômes multiples d'Hermite, on a une relation de récurrence suivante

$$H_{\vec{n}+\vec{e}_i}(x) = (x - \frac{c_i}{2}) H_{\vec{n}}(x) - \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}+\vec{e}_j}(x), \quad 1 \leq i \leq r.$$

Matrices aléatoires et ensemble unitaire gaussien

On considère une matrice hermitienne aléatoire M d'ordre n ayant pour densité de probabilité

$$\frac{1}{Z_n} e^{-\frac{1}{2}\mathrm{Tr}(M^2)} dM$$

Le polynôme caractéristique moyen vérifie

$$\mathbb{E} \det(M - zI_n) = (-1)^n H_n(z)$$

où H_n est une forme probabiliste des polynômes d'Hermite.

Matrices aléatoires avec source extérieure et polynômes multiples d'Hermite

On considère une matrice hermitienne aléatoire M avec source extérieure A

$$\frac{1}{Z_{|\vec{n}|}} e^{-\mathrm{Tr} M^2 - AM)} dM$$

définie sur une matrice hermitienne M d'ordre $|\vec{n}| \times |\vec{n}|$.
Le polynôme caractéristique moyen vérifie

$$\mathbb{E}[\det(M - zI_{|\vec{n}|})] = (-1)^{|\vec{n}|} H_{\vec{n}}(z)$$

où $H_{\vec{n}}$ sont des polynômes multiples d'Hermite.

Problème de Riemann-Hilbert

Le problème de Riemann-Hilbert scalaire et additif consiste à trouver une fonction $f : \mathbb{C} \to \mathbb{C}$ telle que

- f est analytique sur $\mathbb{C} \setminus \mathbb{R}$
- pour $x \in \mathbb{R}$,
$$f_+(x) = f_-(x) + \omega(x)$$
- as $z \to \infty$, $f(z) = O(\frac{1}{z})$

L'unique solution à ce problème est donnée par

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(s)}{s - z} ds$$

Le problème de Riemann-Hilbert pour les polynômes orthogonaux sur l'axe réel est de trouver la matrice fonction $Y : \mathbb{C} \to \mathbb{C}^{2\times 2}$ satisfaisant

- Y est analytique sur $\mathbb{C} \setminus \mathbb{R}$
- Sur l'axe réel, on a
$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}$$
- Y a pour comportement asymptotique
$$Y(z) = \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

Pour $n \geq 1$, la solution de ce problème de RH est donnée par

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{\pi i}\int_{\mathbb{R}} \frac{P_n(s)\omega(s)}{s-z}ds \\ -2\pi i\gamma_{n-1}^2 P_{n-1}(z) & -\gamma_{n-1}^2 \int_{\mathbb{R}} \frac{P_{n-1}(s)\omega(s)}{s-z}ds \end{pmatrix}$$

Considérons par exemple le problème de RH pour le cas des polynômes d'Hermite où $Y : \mathbb{C} \to \mathbb{C}^{2\times 2}$ est une fonction matricielle avec les propriétés suivantes:

- Y est analytique sur $\mathbb{C} \setminus \mathbb{R}$
- Les valeurs bornées Y_+ et Y_- existent sur $\mathbb{R}$ et

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}$$

- Y a pour comportement asymptotique

$$Y(z) = \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Alors

$$Y(z) = \begin{pmatrix} h_n(z) & \frac{1}{\pi i}\int_{\mathbb{R}} \frac{h_n(s)\omega(s)}{s-z}ds \\ -2\pi i\gamma_{n-1}^2 h_{n-1}(z) & -\gamma_{n-1}^2 \int_{\mathbb{R}} \frac{h_{n-1}(s)\omega(s)}{s-z}ds \end{pmatrix}$$

où $h_n = 2^{-n}H_n$ sont des polynômes moniques d'Hermite.

Pour le cas multiple, considérons le problème de RH pour $r = 2$. Il faut alors trouver $Y : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}^{3\times 3}$ telle que

- Y est analytique sur $\mathbb{C} \setminus \mathbb{R}$.
  Une fonction matricielle Y est analytique au point z si si chacune de ses composantes est une fonction analytique au point z.

- $$Y_\pm(x) = \lim_{\epsilon \to 0^\pm} Y(x \pm i\epsilon) \text{ existent et } \forall x \in \mathbb{R},$$

  on a

  $$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1(x) & \omega_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}$$

  où

  $$\omega_1(x) = e^{-|\vec{n}|(V(x)-ax)} \quad \text{et} \quad \omega_2(x) = e^{-|\vec{n}|(V(x)+ax)}$$

- Pour $z \to \infty$, on a un comportement asymptotique

$$Y(z) = \left(I + O(\tfrac{1}{z})\right)\begin{pmatrix} z^{|\vec{n}|} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$$

  où I est une matrice identité d'ordre 3×3.

La solution de ce système existe lorsque le multi-indice $\vec{n} = (n_1, n_2)$ est normal.Elle contient les polynômes orthogonaux multiples de type II $P_{\vec{n}}$ et elle se présente comme suit

$$Y(z) = \begin{pmatrix} P_{\vec{n}}(z) & \frac{1}{2\pi i}\int \frac{P_{\vec{n}}(x)\omega_1(x)}{x-z}dx & \frac{1}{2\pi i}\int \frac{P_{\vec{n}}(x)\omega_2(x)}{x-z}dx \\ -2\pi i\gamma_1 P_{\vec{n}-\vec{e_1}}(z) & -\gamma_1\int \frac{P_{\vec{n}-\vec{e_1}}(x)\omega_1(x)}{x-z}dx & -\gamma_1\int \frac{P_{\vec{n}-\vec{e_1}}(x)\omega_2(x)}{x-z}dx \\ -2\pi i\gamma_2 P_{\vec{n}-\vec{e_2}}(z) & -\gamma_2\int \frac{P_{\vec{n}-\vec{e_2}}(x)\omega_1(x)}{x-z}dx & -\gamma_2\int \frac{P_{\vec{n}-\vec{e_2}}(x)\omega_2(x)}{x-z}dx \end{pmatrix}$$

Notons que problème de RH similaire existe aussi pour polynômes orthogonaux multiples de type I.

Relations de récurrence et équation différentielle

On a les relations de récurrence

$$\psi_{(n_1+1,n_2)}(z) = \begin{pmatrix} z-a & -\frac{n_1}{|\vec{n}|} & -\frac{n_2}{|\vec{n}|} \\ 1 & 0 & 0 \\ 1 & 0 & -2a \end{pmatrix}\psi_{n_1,n_2}(z)$$

$$\psi_{(n_1,n_2+1)}(z) = \begin{pmatrix} z+a & -\frac{n_1}{|\vec{n}|} & -\frac{n_2}{|\vec{n}|} \\ 1 & 2a & 0 \\ 1 & 0 & 0 \end{pmatrix}\psi_{n_1,n_2}(z)$$

et l'équation différentielle

$$\psi'_{(n_1,n_2)}(z) = |\vec{n}|\begin{pmatrix} -z & \frac{n_1}{|\vec{n}|} & \frac{n_2}{|\vec{n}|} \\ -1 & -a & 0 \\ -1 & 0 & a \end{pmatrix}\psi_{n_1,n_2}(z).$$

Pour le cas des polynômes multiples d'Hermite, on a les équations de récurrence

$$P_{(n_1,n_2+1)}(z) = (z - \tilde{b}_{(n_1,n_2)}P_{(n_1,n_2)}(z) - c_{(n_1,n_2)}P_{(n_1-1,n_2)}(z) - d_{(n_1,n_2)}P_{(n_1,n_2-1)}(z)$$

et

$$P_{(n_1-1,n_2+1)}(z) = P_{(n_1,n_2)}(z) - \tilde{e}_{(n_1,n_2)}P_{(n_1-1,n_2)}(z)$$

et l'équation différentielle dans ce cas est

$$\Psi'_{(n_1,n_2)}(z) = |\vec{n}|A_{(n_1,n_2)}(z)\Psi_{(n_1,n_2)}(z).$$

Méthode de descente de Deift-Zhou

Cette méthode permet d'analyser le problème de RH pour $z \to \infty$. Elle consiste à transformer le problème de RH initial Y, dans quelques étapes, en un autre problème de RH équivalent R qui est aussi analytique et normalisé à l'infini. La solution R de ce nouveau modèle du problème de RH convergera à la matrice identité. En inversant ces étapes, on peut enfin trouver la solution exacte du problème original Y.

Comportement asymptotique des polynômes multiples d'Hermite

On partitionne le plan complexe en trois régions et on dérive, dans chaque région, le comportement asymptotique des polynômes multiples d'Hermite.

Région à l'extérieur des lentilles et des disques

Dans cette région, on dérive le polynômes multiples d'Hermite de la forme

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \frac{\xi_1^2 - a^2}{\sqrt{(\xi_1^2 - p^2)(\xi_1^2 - q^2)}} e^{-|\vec{n}|\lambda(z)} \Big(1 + O\Big(\frac{1}{|\vec{n}|(|z|+1)}\Big)\Big).$$

Région à l'intérieur des lentilles mais à l'extérieur des disques

Dans la région se trouvant au dessus de la lentille sur $[z_2, z_1]$, on dérive le polynômes multiples d'Hermite de la forme

$$\begin{aligned} P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} &= \Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_1(z) + |\vec{n}|l_1} \\ &+ \Big(\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_2(z) + |\vec{n}|l_1} \end{aligned}$$

où

$$\lambda_k(z) = \int_{z_1}^{z} \xi_k(s)ds$$

De façon similaire, dans la région se trouvant en dessous de la lentille sur $[z_2, z_1]$, on dérive le polynômes multiples d'Hermite de la forme

$$\begin{aligned} P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} &= \Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_1(z) + |\vec{n}|l_1} \\ &- \Big(\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_2(z) + |\vec{n}|l_1}. \end{aligned}$$

Pour $z = x$ réel, $x \in [z_2 + r, z_1 - r]$, cette dernière équation peut s'écrire en fonction de la densité des valeur propre comme

$$P_{\vec{n}}(x)e^{-\frac{|\vec{n}|}{2}x^2} = A(x)cos[|\vec{n}|\pi \int_{z_1}^{x} \rho(s)ds - \varphi(x)] + O(\frac{1}{|\vec{n}|})e^{-|\vec{n}|Re\lambda_{1+}(x)+|\vec{n}|l_1}.$$

On peut dériver une formule similaire sur l'intervalle $[-z_1 + r, -z_2 - r]$.

Region à l'intérieur des disques

Considérons le disque $D(z_1, r)$. Dans cette régions, on dérive les polynômes multiples d'Hermite en quatre régions.

Région I et IV

Dans cette région, on dérive le polynômes multiples d'Hermite de la forme

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[|\vec{n}|^{1/6}B(z)Ai(|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) + |\vec{n}|^{-1/6}C(z)$$
$$Ai'(|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1}$$

où

$$B(z) = \beta(z)^{1/4}\Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}}\Big)$$

et

$$C(z) = \beta(z)^{-1/4}\Big(- \frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}}\Big).$$

Région II

Dans cette région, on dérive le polynômes multiples d'Hermite de la forme

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[- \omega|\vec{n}|^{1/6}B(z)Ai(\omega|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) - \omega^2|\vec{n}|^{-1/6}C(z)$$
$$Ai'(\omega|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1}$$

où $B(z)$ et $C(z)$ sont définies dans les régions I et IV.

Région III

De façon similaire, dans cette région, on dérive le polynômes multiples d'Hermite de la forme

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[- \omega|\vec{n}|^{1/6}B(z)Ai(\omega^2|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) - \omega^4|\vec{n}|^{-1/6}C(z)$$
$$Ai'(\omega^2|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1}$$

où $B(z)$ et $C(z)$ sont définies dans les régions I et IV.

Conclusion

Revisitons les résultats principaux de ce travail. Pour cela rappelons que l'objectif de ce travail était d'expliciter les différentes propriétés des polynômes multiples d'Hermite, montrer la relation avec les matrices aléatoires avec source extérieure et d'analyser le comportement asymptotique de ces polynômes en utilisant la méthode de descente de Deif-Zhou basée sur le problème de RH pour $z \to \infty$.
Nous rappelons en premier lieu les polynômes orthogonaux classiques, les polynômes orthogonaux multiples et nous prouvons certaines propriétés pour le cas d'Hermite.Pour cela, nous donnons deux autres démonstrations pour trouver la fonction génératrice des polynômes multiples d'Hermite et nous donnons la forme correcte du Théorème 2.4 de Lee qui devient le Théorème 3.0.3 de ce travail.

Contents

Introduction

Throughout this work, r is assumed to be a positive integer. We will use a multi-index $\vec{n} = (n_1, n_2, ..., n_r)$ and its length $|\vec{n}| = n_1 + n_2 + ... + n_r$.

Multiple Hermite polynomials $H_{\vec{n}}$ are polynomials that satisfy orthogonality relations [10, Chapter 23]

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k w_j(x) dx = 0 \quad k = 0, 1, ...n_j - 1, \qquad j = 1, ...r$$

where w_j is the Hermite weigth defined by

$$w_j(x) = e^{-x^2 + c_j x}, \quad j = 1, ..., r$$

for the physics form.
One can write the probabilistic form of the Hermite weight as

$$w_j(x) = e^{\frac{-1}{2}x^2 + c_j x}, \quad j = 1, ..., r$$

or

$$w_j(x) = e^{\frac{\delta}{2}x^2 + c_j x}, \quad j = 1, ..., r \quad \delta < 0$$

for example in [2] and [13] respectively. In this work we will use

$$w_j(x) = e^{-x^2 + c_j x}, \quad j = 1, ..., r.$$

In the first part we prove the Rodrigues formula, the explicit expression, the properties of derivatives,recurrence relations,integral representation and the generating function of multiple Hermite polynomials and we introduce the Hermite-Padé approximation.
Consider a random matrix ensemble with external source

$$\frac{1}{Z_n} e^{-Tr(V(M) - AM)} dM \tag{1}$$

defined on $n \times n$ Hermitian matrices M, where

$$V : \mathbb{R} \to \mathbb{R}$$

is a function with enough increase at $\pm\infty$ such that the integral

$$Z_n = \int e^{-Tr(V(M) - AM)} dM$$

converges. The case of multiple Hermite polynomials correspond to the random matrix model (1) with

$$V(M) = \frac{1}{2} M^2.$$

We then give the average characteristic polynomials and we extend at large n limit of Gaussian random matrices with external source, where we consider the random matrix ensemble

$$\mu(dM) = \frac{1}{Z_n} e^{-nTr(V(M)-AM)} dM.$$

In this part we will describe a Riemann-Hilbert problem and its transformations,recurrence relations,differential equations of multiple orthogonal in general and in particular for the case of Multiple Hermite polynomials.

Chapter 1

Multiple orthogonal polynomials

For this dicussion we refer to the references [18],[10] and [19].

1.1 Classical orthogonal polynomials

Consider the differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$$

where σ is a polynomial of degree at most 2, τ a polynomial of degree 1 and λ is a constant. Orthogonal polynomials are polynomials that satisfy

$$\int_a^b p_n(x)p_m(x)d\mu(x) = \delta_{mn}, \quad m, n \geq 0 \tag{1.1}$$

where μ is a positive measure on $[a, b]$ such that

$$d\mu(x) = w(x)dx$$

and ω is a solution of Pearson equation

$$(\sigma w)' = \tau w$$

with

$$(\sigma w)(a) = 0 = (\sigma w)(b)$$

Orthogonal polynomials always satisfy a three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

with $n \geq 0,\ p_{-1} = 0,\ p_0 = \frac{1}{\sqrt{m_0}},\ m_0 = \int d\mu(x)$
where

$$a_n = \int_a^b xp_n(x)p_{n-1}(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0$$

and

$$b_n = \int_a^b x p_n^2(x) d\mu(x)$$

γ_n is the leading coefficient of the orthogonal polynomial p_n. Monic orthogonal polynomials on the real line $P_n(x) = \frac{p_n(x)}{\gamma_n}$ satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x)$$

$$n \geq 0,\ P_{-1} = 0,\ P_0 = 1.$$

The recurrence coefficients give important information about the orthogonality measure μ.

The Rodrigues formula for orthogonal polynomials is

$$w(x)p_n(x) = c_n \frac{d^n}{dx^n}(w(x)\sigma^n(x)) \tag{1.2}$$

where c_n is a constant of normalization.

Summary of the classical orthogonal polynomials:Hermite, Laguerre and Jacobi

	Hermite $H_n(x)$	Laguerre $L_n^{\alpha}(x)$	Jacobi $P_n^{(\alpha,\beta)}(x)$
interval	$(-\infty, \infty)$	$[0, \infty)$	$[-1, 1]$
$w(x)$	e^{-x^2}	$x^{\alpha}e^{-x}$	$(1-x)^{\alpha}(1+x)^{\beta}$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-x(\alpha+\beta+2)-\alpha+\beta$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
c_n	$(-1)^n$	$\frac{1}{n!}$	$\frac{(-1)^n}{2^n n!}$
Remark	$-$	$\alpha > -1$	$\alpha, \beta > -1$

Table 1.1: Classical orthogonal polynomials.

Legendre polynomials are Jacobi polynomials with $\alpha = \beta = 0$.

Let μ be a positive measure with moments

$$m_n = \int_{\mathbb{R}} x^n d\mu(x).$$

The n^{th} degree monic orthogonal polynomial P_n is defined by requiring that

$$\int_{\mathbb{R}} P_n(x)x^k d\mu(x) = 0, k = 0, 1, ..., n-1 \tag{1.3}$$

and the n^{th} degree orthonormal polynomial $p_n = \gamma_n P_n$ is defined by taking γ_n from

$$\int_{\mathbb{R}} P_n(x) x^n d\mu(x) = \gamma_n^{-2}, \quad \gamma_n > 0.$$

The system (1.3) is a linear system of n equations for n unknown coefficients $a_{k,n}(k = 1, ..., n)$ of the monic polynomial

$$P_n(x) = \sum_{k=0}^{n} a_{k,n} x^{n-k}$$

with

$$a_{0,n} = 1.$$

1.2 Definitions: type I and type II multiple orthogonal polynomials

Multiple orthogonal polynomials are polynomials of one variable which are defined by orthogonality relations with respect to r different measures $\mu_1, \mu_2, \ldots, \mu_r$ where $r > 1$ is a positive integer.
These polynomials should not be confused with multivariate or multivariable orthogonal polynomials of several variables.

We denote by $r \geq 1$ the number of measures $(\mu_1, \mu_2, ..., \mu_r)$ for the orthogonality conditions. We only consider positive measures on the real line for which all the moments exist.Let $\vec{n} = (n_1, n_2, ...n_r) \in \mathbb{N}^r$ be a multi-index and denote by $|\vec{n}| = n_1 + n_2 + ... + n_r$.

Definition 1.2.1. Type I multiple polynomials for the multi-index $\vec{n}$ consist of the vector $(A_{\vec{n},1}, ..., A_{\vec{n},r})$ for r polynomials, where $A_{\vec{n},j}$ has degree at most $n_j - 1$ for $1 \leq j \leq r$, for which

$$\int x^k \sum_{j=1}^{r} A_{\vec{n},j}(x) d\mu_j(x) = 0, \quad 0 \leq k \leq |\vec{n}| - 2 \tag{1.4}$$

with normalisation

$$\int x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x) d\mu_j(x) = 1. \tag{1.5}$$

We denote $Q_{\vec{n}}$ by

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) \omega_j(x)$$

and

$$d\mu_j(x) = \omega_j(x) dx.$$

Equations (1.4)-(1.5) give a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of $A_{\vec{n},j}(1 \leq j \leq r)$. If this system has a unique solution, then we say that the multi-index $\vec{n}$ is normal.Note that the matrix linear system contains moments of the measures $\mu_1, ..., \mu_r$.

Definition 1.2.2. The type II multiple orthogonal polynomials for the multi-index $\vec{n}$ are the monic polynomials $P_{\vec{n}}$ of degree $\leq\mid \vec{n} \mid$ for which

$$\int P_{\vec{n}}(x)x^k d\mu_j(x) = 0, \quad 0 \leq k \leq n_j - 1, \quad \text{for } 1 \leq j \leq r. \tag{1.6}$$

Equations (1.6) and the normalization $P_{\vec{n}} = x^{|\vec{n}|} + ...$ give a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of $P_{\vec{n}}$.Note that the matrix of this linear system is the transpose of the matrix for type I multiple orthogonal polynomials.So the type II multiple orthogonal polynomial exists and is unique whenever $\vec{n}$ is a normal index.

Definition 1.2.3. A system of measures $(\mu_1, ..., \mu_r)$ for which all multi-indices are normal is called a perfect system [8].

1.3 Special systems

Not all systems of measures $(\mu_1, ..., \mu_r)$ on the real line lead to a perfect system. There are, however, some special systems for which all multi-indices are normal.

Definition 1.3.1. The measures $(\mu_1, ..., \mu_r)$ are an Angelesco system if the supports of the measures $\text{supp}(\mu_j) \subset\triangle_j$ are on disjoint intervals $\triangle_j$,i.e. $\triangle_i \bigcap \triangle_j = \emptyset$ whenever $i \neq j$.

An important property of Angelesco systems is that the type II multiple orthogonal polynomial $P_{\vec{n}}$ has exactly n_j distinct zeros on the open interval $\triangle_j^\circ$ for every $1 \leq j \leq r$, giving a total of $\mid \vec{n} \mid$ real and simple zeros.This implies that every multi-index is normal and that an Angelesco system is perfect.

Definition 1.3.2. The measures $(\mu_1, ..., \mu_r)$ are an AT system(algebraic Chebyshev system) on the interval $[a, b]$ if the measures are all absolutely continuous with respect to a positive measure μ on $[a, b]$,

$$d\mu_j(x) = \omega_j(x)dx,\ 1 \leq j \leq r$$

where ω_j are positive weight functions, and for every $\vec{n}$ the functions

$$\omega_1(x), x\omega_1(x), ...x^{n_1-1}\omega_1(x), \omega_2(x), x\omega_2(x), ...x^{n_2-1}\omega_2(x), ..., \omega_r(x), x\omega_r(x), ...x^{n_r-1}\omega_r(x)$$

are a Chebyshev system on $[a, b]$, i.e. any linear combination of these functions has at most $|\vec{n}|-1$ zeros on $[a, b]$.

For an AT system the type I multiple orthogonal polynonmials are such that the linear combination

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x)\omega_j(x)$$

has exactly $|\vec{n}| - 1$ zeros on the open interval (a, b).Furthermore the type II multiple orthogonal polynomial has $|\vec{n}|$ simple zeros on (a, b). This implies that every multi-index $\vec{n}$ is a normal index.

1.4 Biorthogonality

In an AT system every measure μ_j is absolutely continuous with respect to a given measure μ on [a,b] and $d\mu_j(x) = \omega_j(x)dx, \quad 1 \le j \le r.$
In an Angelesco system we can define $\mu = \mu_1 + \mu_2 + ... + \mu_r$ and if the intervals $[a_k, b_k]$are disjoint, then each measure μ_j is absolutely continuous with respect to μ and $d\mu_j(x) = \omega_j(x)dx$ and then the weight ω_j is the characteristic function for the interval $\triangle_j$

$$\omega_j(x) = \begin{cases} 1, & x \in [a_j, b_j] \\ 0, & x \in \mathbb{R} \setminus [a_j, b_j]. \end{cases}$$

We then define the type I functions by

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x)\omega_j(x). \tag{1.7}$$

The orthogonality conditions (1.4)-(1.5) for type I multiple orthogonal polynomials then become

$$\int x^k Q_{\vec{n}}(x)d\mu(x) = 0, \quad 0 \le k \le |\vec{n}|{-}2$$

$$\int x^{|\vec{n}|-1} Q_{\vec{n}}(x)d\mu_j(x) = 1.$$

Combining this with the orthogonality relations for the type II multiple orthogonal polynomials

$$\int P_{\vec{n}}(x)x^k d\mu_j(x) = 0, \quad 0 \le k \le n_j - 1$$

then gives biorthogonality relations

$$\int P_{\vec{n}}(x)Q_{\vec{m}}(x)d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \le \vec{n} \\ 0, & \text{if } |\vec{n}| \le |\vec{m}|{-}2 \\ 1, & \text{if } |\vec{n}| = |\vec{m}|{-}1 \end{cases}. \tag{1.8}$$

This does not imply that all the $P_{\vec{n}}$ are biorthogonal to all $Q_{\vec{m}}$, since (1.8) does not include all the ways to combine $(\vec{n}, \vec{m})$.

1.5 Recurrence relations

There are a number of recurrence relations for multiple orthogonal polynomials of type I and type II.
The most important recurrence relations are those relating the polynomials with multi-index $\vec{n}$ to the neighboring multi-indices $\vec{n} \pm \vec{e}_i$, where $\vec{e}_i = (0, ..., 0, 1, 0, ...0)$ is the ith standard unit vector in $\mathbb{N}^r$.
The nearest neighbor recurrence relations for the type II multiple orthogonal polynomials are [10, chapter 23] and [19]

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \quad j = 1, 2, ..., r. \tag{1.9}$$

Note that there are r equations and $2r$ recurrence coefficients $(a_{\vec{n},1}, ..., a_{\vec{n},r})$ and $(b_{\vec{n},1}, ..., b_{\vec{n},r})$ for every multi-index $\vec{n}$.
These recurrence relations are valid only when all multi-indices are normal.
The recurrence coefficients are given by [10, chapter 23],[19]

$$a_{\vec{n},j} = \frac{\int x^{n_j} P_{\vec{n}}(x) d\mu_j(x)}{\int x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x) d\mu_j(x)} \tag{1.10}$$

and

$$b_{\vec{n},j} = \int x P_{\vec{n}}(x) Q_{\vec{n}+\vec{e}_j}(x) d\mu(x). \tag{1.11}$$

where $Q_{\vec{n}}$ is the type I function given in (1.7).
There are similar recurrence relations for the type I multiple orthogonal polynomials:

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_k}(x) + b_{\vec{n}-\vec{e}_k,k}Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x), \quad j = 1, 2, ..., r. \tag{1.12}$$

Observe that the same recurrence coefficients $(a_{\vec{n},1}, ..., a_{\vec{n},r})$ are used but that there is a shift in the other recurrence coefficients $(b_{\vec{n}-\vec{e}_1,1}, ..., b_{\vec{n}-\vec{e}_r,r})$.

1.6 Hermite-Padé approximation

Multiple orthogonal polynomials originate from Hermite-Padé approximation where the type II polynomials appear as the common denominators when one wants to approximate r functions simultaneously by rational functions. Hermite used this construction for the first time for his proof that e is a transcendental number, for which he used Hermite-Padé approximation at $z = 1$ to the functions $e^z, e^{2z}, ..., e^{rz}$.
We will use here only Hermite-Padé approximation at infinity to r functions $f_1, f_2, ..., f_r$ of the form

$$f_j(z) = \int \frac{1}{z-x} d\mu_j(x), \quad 1 \leq j \leq r,$$

where $\mu_j (1 \leq j \leq r)$ are positive measures on the real line.
There are two types of Hermite-Padé approximation.

Definition 1.6.1. Type I Hermite-Padé approximation at infinity to $(f_1, f_2, ..., f_r)$ consists of finding r polynomials $A_{\vec{n},j}$ of degree $n_j - 1 (1 \leq j \leq r)$ and a polynomial $B_{\vec{n}}$ such that

$$\sum_{j=1}^{r} A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = O(\frac{1}{z^{|\vec{n}|}}), \quad z \to \infty.$$

The polynomials $(A_{\vec{n},1}, ..., A_{\vec{n},r})$ are then precisely the type I multiple orthogonal polynomials for the system of measures $(\mu_1, ..., \mu_r)$ and

$$B_{\vec{n}}(z) = \int \sum_{j=1}^{r} \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z - x} d\mu_j(x).$$

The error in type I Hermite-Padé approximation is given by

$$\sum_{j=1}^{r} A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = \int \sum_{j=1}^{r} \frac{A_{\vec{n},j}(x)}{z - x} d\mu_j(x).$$

Definition 1.6.2. Type II Hermite-Padé approximation at infinity to $(f_1, f_2, ..., f_r)$ consists of finding a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}|$ and polynomials $Q_{\vec{n},j}$ such that

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = O(\frac{1}{z^{n_j+1}}), \quad z \to \infty$$

for every j such that $1 \leq j \leq r$.

The common denominator $P_{\vec{n}}$ is then the type II multiple orthogonal polynomial for the system of measures $(\mu_1, ..., \mu_r)$ and

$$Q_{\vec{n},j}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z - x} d\mu_j(x), \quad 1 \leq j \leq r.$$

The error in type II Hermite-Padé approximation is then given by

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = \int \frac{P_{\vec{n}}(x)}{z - x} d\mu_j(x), \quad 1 \leq j \leq r.$$

Chapter 2

Multiple Hermite polynomials

In this chapter, we refer to the references [14],[18],[10],[4] and[1].

2.1 Classical Hermite polynomials

Consider the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{2.1}$$

The polynomial solution of (2.1) is a Hermite polynomial H_n of degree n.
Hermite polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{mn}.$$

The Rodrigues formula for this polynomial is given by

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}). \tag{2.2}$$

The three-term recurrence relation is given by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{2.3}$$

One has the raising operators of the form [10]

$$(e^{-x^2}H_{n-1}(x))' = -e^{-x^2}H_n(x)$$

and the lowering operators of the form

$$(H_n(x))' = 2nH_{n-1}(x).$$

The Hermite polynomials have the explicit expression :

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}. \tag{2.4}$$

The first Hermite polynomials are:
$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$
$H_4(x) = 16x^4 - 48x^2 + 12$

Observe that $H_n(x) = 2^n x^n + \dots$

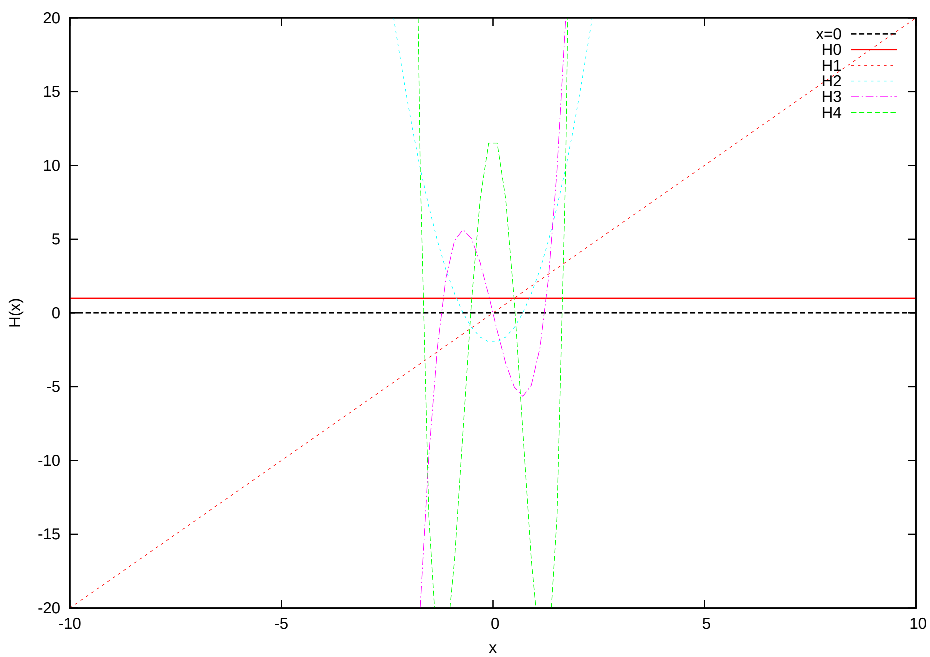

Figure 2.1: The first Hermite polynomials

Hermite polynomials are neither normalized nor monic but another normalization is used.

The generating function has the form

$$G(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.$$

By the recurrence relation in (2.3), one has

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_{n+1}(x)}{n!}t^n &= 2x\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n - 2\sum_{n=0}^{\infty}\frac{nH_{n-1}(x)}{n!}t^n\\ &= 2xG(x,t) - 2tG(x,t).\end{aligned}$$

We have also

$$\frac{\partial G}{\partial t} = \sum_{n=1}^{\infty}\frac{H_n(x)n}{n!}t^{n-1} = \sum_{n=0}^{\infty}\frac{H_{n+1}(x)}{n!}t^n.$$

Then

$$\frac{\partial G}{\partial t} = 2(x-t)G(x,t)$$

so that

$$G(x,t) = e^{c(x)}e^{2xt-t^2}$$

Where $G(x,0) = e^{c(x)} = 1$.
Finally, the Hermite polynomials have a generating function of the form

$$G(x,t) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = e^{2xt-t^2} \tag{2.5}$$

We have

$$\left(e^{-x^2}H_n(x)\right)^{(k)} = (-1)^k e^{-x^2}H_{n+k}(x)$$

and then (2.5) becomes

$$e^{-(x-t)^2} = \sum_{n=0}^{\infty}e^{-x^2}\frac{H_n(x)}{n!}t^n$$

so that

$$\frac{d^k}{dx^k}e^{-(x-t)^2} = \sum_{n=0}^{\infty}(-1)^k e^{-x^2}\frac{H_{n+k}(x)}{n!}t^n$$

then

$$e^{x^2}(-1)^k e^{-(x-t)^2} = \sum_{n=0}^{\infty}\frac{H_{n+k}(x)}{n!}t^n. \tag{2.6}$$

By the Rodrigues formula, one has

$$e^{-x^2}H_n(x-t) = (-1)^n\frac{d^n}{dx^n}e^{-(x-t)^2}.$$

The left member of equation (2.6) becomes

$$e^{x^2}e^{-(x-t)^2}H_k(x-t) = e^{2tx-t^2}H_k(x-t).$$

Finally, (2.6) generalize the generating function in the form

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x)}{n!} t^n = e^{2xt-t^2} H_k(x-t). \tag{2.7}$$

From (2.5) if we put $x = cx$ and $t = t/c$, (2.5) becomes

$$e^{2tx-t^2/c^2} = \sum_{n=0}^{\infty} \frac{H_n(cx)}{n!} t^n c^{-n} \tag{2.8}$$

But

$$\begin{aligned}
e^{2tx-t} e^{t^2-t^2/c^2} &= \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \sum_{n=0}^{\infty} (1-\frac{1}{c^2})^k \frac{t^{2k}}{k!} \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{(\frac{c^2-1}{c^2})^{\frac{k}{2}}}{\frac{k}{2}!} \frac{H_{n-k}(x)}{(n-k)!}, \text{ with } k \text{ odd} \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]} \frac{(\frac{c^2-1}{c^2})^k}{k!} \frac{H_{n-2k}(x)}{(n-2k)!}.
\end{aligned} \tag{2.9}$$

Comparing (2.8) and (2.9), one has the expansion of scaled Hermite polynomials

$$H_n(cx) = \sum_{n=0}^{[\frac{n}{2}]} \frac{n!(-1)^k}{k!(n-2k)!} (1-c^2)^k c^{n-2k} H_{n-2k}(x). \tag{2.10}$$

2.2 Type I and type II multiple Hermite polynomials

Multiple Hermite polynomials play an important role in analyzing random matrices with external source and also in the analysis of non-intersecting-Brownian motions leaving from one point and arriving at r distinct points. But in this work, we focus on random matrices with external source.

r is assumed to be a positive integer. We use a multi-index $\vec{n} = (n_1, n_2, ..., n_r)$ and its length $|\vec{n}| = n_1 + n_2 + ... + n_r$ and $c_1, c_2, ..., c_r$ are r different real parameters such that $c_i \neq c_j$ whenever $i \neq j$.

Definition 2.2.1. Multiple Hermite polynomials of type I for the multi-index $\vec{n}$ consist of the vector $(A_{\vec{n},1}, ..., A_{\vec{n},r})$ of r polynomials, where $A_{\vec{n},j}$ has degree at most $n_j - 1$ for $1 \leq j \leq r$, such that

$$\int_{-\infty}^{+\infty} x^k \sum_{j=1}^{r} A_{\vec{n},j}(x) e^{-x^2+c_j x} dx = 0, \quad 0 \leq k \leq |\vec{n}|-2 \tag{2.11}$$

with normalisation

$$\int_{-\infty}^{+\infty} x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x) e^{-x^2+c_j x} dx = 1. \tag{2.12}$$

where $c_i \neq c_j$ whenever $i \neq j$.

Definition 2.2.2. The multiple Hermite polynomial of type II for the multi-index $\vec{n}$ is the monic polynomial $H_{\vec{n}}$ of degree $\leq |\vec{n}|$ for which

$$\int_{-\infty}^{+\infty} H_{\vec{n}}(x) x^k e^{-x^2+c_j x} dx = 0, \quad 0 \leq k \leq n_j - 1 \tag{2.13}$$

for $1 \leq j \leq r$.
Some other authors denote multiple Hermite polynomials as $H_{\vec{n}}^{\vec{c}}$.

2.3 Rodrigues formula

The Rodrigues formula of the multiple Hermite polynomials is given by

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \left(\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}\right) e^{-x^2} \tag{2.14}$$

We can prove that $H_{\vec{n}}$ in (2.14) is the multiple Hermite polynomial.
In fact, we rewrite (2.14) as

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \left(\prod_{j=1}^{r} D_j\right) e^{-x^2} \tag{2.15}$$

where

$$D_j = e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}. \tag{2.16}$$

We show that $(D_1, \ldots, D_r)$ defined by (2.16) commute.
We have

$$\begin{aligned}
(D_i D_j) f &= e^{-c_i x} \frac{d^{n_i}}{dx^{n_i}} e^{c_i x} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} (e^{c_j x} f) \\
&= e^{-c_i x} \frac{d^{n_i}}{dx^{n_i}} e^{c_i x} \sum_{k=0}^{n_j} \binom{n_j}{k} f^{(k)}(x) c_j^{n_j-k} e^{c_j x} \\
&= e^{-c_i x} \frac{d^{n_i}}{dx^{n_i}} (e^{c_i x} f^{(k)}(x)) \sum_{k=0}^{n_j} \binom{n_j}{k} c_j^{n_j-k} e^{c_j x} \\
&= e^{-c_i x} \sum_{l=0}^{n_i} \binom{n_i}{l} \sum_{k=0}^{n_j} \binom{n_j}{k} c_i^{n_i-l} c_j^{n_j-k} f^{(k+l)}(x) e^{c_i x} \\
&= \sum_{l=0}^{n_i} \sum_{k=0}^{n_j} \binom{n_i}{l} \binom{n_j}{k} c_i^{n_i-l} c_j^{n_j-k} f^{(k+l)}(x).
\end{aligned} \tag{2.17}$$

Similarly

$$
\begin{aligned}
(D_j D_i) f &= \sum_{l=0}^{n_j} \sum_{k=0}^{n_i} \binom{n_j}{l} \binom{n_i}{k} c_j^{n_j - l} c_i^{n_i - k} f^{(k+l)}(x) \\
&= \sum_{k=0}^{n_j} \sum_{l=0}^{n_i} \binom{n_j}{k} \binom{n_i}{l} c_j^{n_j - k} c_i^{n_i - l} f^{(k+l)}(x). \qquad (2.18)
\end{aligned}
$$

Comparing (2.17) and (2.18) it follows that $(D_1, \ldots, D_r)$ commute.

We can then rewrite the equation (2.15):

$$
\begin{aligned}
\int_{-\infty}^{\infty} H_{\vec{n}}(x) e^{-x^2 + c_j x} x^k dx &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{-\infty}^{\infty} \Big[D_j \prod_{i \neq j} D_i e^{-x^2} \Big] e^{c_j x} x^k dx \\
&= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{-\infty}^{\infty} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x} \prod_{i \neq j} D_i e^{-x^2} e^{c_j x} x^k dx.
\end{aligned}
$$

By integration by parts, one has

$$
\int_{-\infty}^{\infty} H_{\vec{n}}(x) e^{-x^2 + c_j x} x^k dx = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{-\infty}^{\infty} \Big[e^{c_j x} \Big(\prod_{i \neq j} D_i \Big) e^{-x^2} \Big(\frac{d^{n_j}}{dx^{n_j}} x^k \Big) \Big] dx \qquad (2.19)
$$

But $\frac{d^{n_j}}{dx^{n_j}} x^k = 0 \qquad \text{for} \qquad 0 \leq k \leq n_j - 1$
then

$$
\int_{-\infty}^{\infty} H_{\vec{n}}(x) e^{-x^2 + c_j x} x^k dx = 0, \qquad 0 \leq k \leq n_j - 1, \qquad j = 1, \ldots, r \qquad (2.20)
$$

which proves that $H_{\vec{n}}$ is the multiple Hermite polynomial.

2.4 Explicit expression

We first establish the explicit expression for $r = 2$ and we generalize for all $j = 1, ..., r$. Using the Rodrigues formula defined by (2.14), if we write it for $r = 2$ and using the Leibniz rule for the 2^{nd} derivative of a product, one has

$$
e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} (e^{-c_1 x} \frac{d^{n_1}}{dx^{n_1}} e^{c_1 x} . e^{-c_2 x} \frac{d^{n_2}}{dx^{n_2}} e^{c_2 x}) e^{-x^2}
$$

Note that $e^{-c_k x} \dfrac{d^{n_k}}{dx^{n_k}} e^{c_k x}$ are the weighted Rodrigues operators for $k = 1, 2, ..., r$ and they commute.

Then

$$\begin{aligned} e^{-x^2} H_{\vec{n}}(x) &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} (e^{-c_1 x} \frac{d^{n_1}}{dx^{n_1}} e^{c_1 x}) \sum_{k_2=0}^{n_2} \binom{n_2}{k_2} c_2^{n_2-k_2} (-2x)^{k_2} e^{-x^2} \\ &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} c_1^{n_1-k_1} c_2^{n_2-k_2} (-1)^{k_1+k_2} (2x)^{k_1+k_2} e^{-x^2}. \end{aligned}$$

For $r = 2$, the explicit expression for multiple Hermite polynomials is given by

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1} \binom{n_2}{k_2} c_1^{n_1-k_1} c_2^{n_2-k_2} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x)$$

with $|\vec{k}| = k_1 + k_2$ and $H_{|\vec{k}|}(x) = 2^{|\vec{k}|} x^{|\vec{k}|} + ...$ is the usual Hermite polynomial of degree $|\vec{k}|$.
For each $0 \leq k_j \leq n_j$, $(j = 1, \ldots, r)$, multiple Hermite polynomials have the following explicit expression

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} c_1^{n_1-k_1} \cdots c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x). \tag{2.21}$$

Recall that $H_{\vec{n}}$ is a monic polynomial of degree $|\vec{k}| = k_1 + k_2 + \ldots + k_r$ and

$$H_{\vec{n}}(x) = x^{|\vec{n}|} - \frac{1}{2} \sum_{j=1}^{r} n_j c_j x^{|\vec{n}|-1} + \ldots$$

Example 2.4.1. For $r = 2$, according to the explicit expression of multiple Hermite polynomials, we have

$$\begin{aligned} H_{0,0}(x) &= 1 \equiv H_0(x), \\ H_{1,0}(x) &= x - \frac{c_1}{2}, \\ H_{0,1}(x) &= x - \frac{c_2}{2}, \\ \cdots \end{aligned}$$

2.5 Properties of derivatives

By the Rodrigues formula defined by (2.14), one has

$$e^{-x^2+c_j x} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} (\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}) e^{-x^2+c_j x} \tag{2.22}$$

so that

$$e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x) = \frac{(-1)^{|\vec{n}-\vec{e}_j|}}{2^{|\vec{n}-\vec{e}_j|}}(\prod_{j=1}^{r} e^{-c_jx}\frac{d^{n_j-1}}{dx^{n_j-1}}e^{c_jx})e^{-x^2+c_jx}.$$

Hence

$$\frac{d}{dx}(e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x)) = -2\frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}}(\prod_{j=1}^{r} e^{-c_jx}\frac{d^{n_j}}{dx^{n_j}}e^{c_jx})e^{-x^2+c_jx}.$$

Since (2.22), multiple Hermite polynomials have the r raising operators of the form

$$\frac{d}{dx}(e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x)) = -2e^{-x^2+c_jx}H_{\vec{n}}(x), \quad 1 \le j \le r \tag{2.23}$$

The self-adjoint form of multiple Hermite polynomials is given by

$$\frac{d}{dx}(e^{-x^2+c_jx}H'_{\vec{n}}(x)) + 2\sum_{j=1}^{r} n_j e^{-x^2+c_jx}H_{\vec{n}}(x) = 0$$

so that

$$\frac{d}{dx}(e^{-x^2+c_jx}H'_{\vec{n}}(x)) - \sum_{j=1}^{r} n_j(-2e^{-x^2+c_jx}H_{\vec{n}}(x)) = 0. \tag{2.24}$$

Using (2.23) in (2.24), one has

$$\frac{d}{dx}(e^{-x^2+c_jx}H'_{\vec{n}}(x)) - \sum_{j=1}^{r} n_j\frac{d}{dx}(e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x)) = 0.$$

After integration by parts, one has

$$e^{-x^2+c_jx}H'_{\vec{n}}(x) - \sum_{j=1}^{r} n_j e^{-x^2+c_jx}H_{\vec{n}-\vec{e}_j}(x) = const.$$

For $x \to \pm\infty$, $\quad const = 0$ and thus multiple Hermite polynomials have the lowering operator of the form

$$\frac{d}{dx}(H_{\vec{n}}(x)) = \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x). \tag{2.25}$$

Consider the differential operators $D = \frac{d}{dx}, \quad D_j = e^{x^2-c_jx}De^{-x^2+c_jx}$
then $D_1, D_2, ..., D_r$ are commuting operators. Combining the raising operators and the lowering operator defined by (2.23) and (2.25), we obtain the differential equation of order $r+1$.
In fact, By the raising operator, one has

$$D(e^{-x^2+c_jx}H_{\vec{n}}(x)) = -2e^{-x^2+c_jx}H_{\vec{n}+\vec{e}_j}(x)$$

so that

$$D_j H_{\vec{n}}(x) = -2H_{\vec{n}+\vec{e}_j}(x).$$

The lowering operator implies that

$$DH_{\vec{n}+\vec{e}_j}(x) = \sum_{j=1}^{r} n_j H_{\vec{n}}(x).$$

As $(D_1, \ldots, D_r)$ commute, one has

$$\begin{aligned} D_j DH_{\vec{n}}(x) &= -2DH_{\vec{n}+\vec{e}_j}(x) \\ &= -2\sum_{j=1}^{r} n_j H_{\vec{n}}(x). \end{aligned}$$

And thus

$$(D_1 \ldots D_j \ldots D_r)H_{\vec{n}}(x) = -2\sum_{j=1}^{r} n_j \prod_{i\neq j} D_i H_{\vec{n}}(x).$$

Finally, we have the following differential equation

$$\Big(\prod_{j=1}^{r} D_j\Big) DH_{\vec{n}}(x) = -2\Big(\sum_{j=1}^{r} n_j \prod_{i\neq j} D_i\Big) H_{\vec{n}}(x). \tag{2.26}$$

2.6 Recurrence relations

Recall that multiple Hermite polynomials of type II $H_{\vec{n}}$ are monic polynomials of degree $|\vec{n}|$ and

$$\int_{-\infty}^{+\infty} H_{\vec{n}}(x) x^k e^{-x^2+c_j} dx = 0, \quad k = 0, 1, ..., n_j - 1, \quad j = 1, 2, ..., r. \tag{2.27}$$

In this section, we refer to the Section 1.5. Using the Rodrigues formula defined by (2.14) one has

$$g_r(t) = \int_{\mathbb{R}} e^{tx} H_{\vec{n}}(x) e^{-x^2} dx = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{\mathbb{R}} e^{tx} \Big(\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}\Big) e^{-x^2} dx. \tag{2.28}$$

We denote

$$D_j = e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}$$

and then

$$D_j f = e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} (e^{c_j x} f).$$

Using the fact that $(D_1, \ldots, D_r)$ commute, (2.28) becomes

$$\begin{aligned} g_r(t) &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{\mathbb{R}} e^{tx} e^{-c_r x} \frac{d^{n_r}}{dx^{n_r}} e^{c_r x} \Big(\prod_{j=1}^{r-1} D_j \Big) e^{-x^2} dx \\ &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \int_{\mathbb{R}} e^{(t-c_r)x} \frac{d^{n_r}}{dx^{n_r}} \Big[e^{c_r x} \Big(\prod_{j=1}^{r-1} D_j \Big) \Big] e^{-x^2} dx. \end{aligned} \tag{2.29}$$

By integration by parts r times, one has

$$\begin{aligned} g_r(t) &= \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} (-1)^{n_r} (t-c_r)^{n_r} \int_{\mathbb{R}} e^{tx} \Big(\prod_{j=1}^{r-1} D_j \Big) e^{-x^2} dx \\ &= \frac{(t-c_r)^{n_r}}{2^{n_r}} \frac{(-1)^{n_1+\ldots+n_r-1}}{(2)^{n_1+\ldots+n_r-1}} \int_{\mathbb{R}} e^{tx} \Big(\prod_{j=1}^{r-1} D_j \Big) e^{-x^2} dx \\ &= \frac{(t-c_r)^{n_r}}{2^{n_r}} g_{r-1}(t). \end{aligned} \tag{2.30}$$

By induction, one has

$$g_r(t) = \frac{(t-c_r)^{n_r} (t-c_{r-1})^{n_r-1}}{2^{n_r} 2^{n_r-1}} \ldots g_0(t)$$

where

$$g_0(t) = \int_{\mathbb{R}} e^{tx} e^{-x^2} dx = \int_{\mathbb{R}} e^{-x^2+tx} dx = \int_{\mathbb{R}} e^{-(x-\frac{t}{2})^2 + t^2/4} dx = e^{t^2/4} \sqrt{\pi}.$$

Finally, we obtain

$$g_r(t) = \int_{\mathbb{R}} e^{tx} H_{\vec{n}}(x) e^{-x^2} dx = 2^{-|\vec{n}|} \sqrt{\pi} (t-c_1)^{n_1} ... (t-c_r)^{n_r} e^{t^2/4}, \tag{2.31}$$

and hence

$$\int_{\mathbb{R}} x^k H_{\vec{n}}(x) e^{-x^2 + c_j(x)} dx = g^{(k)}(c_j) = 0, \quad k = 0, 1, ..., n_j - 1 \text{ for } \quad 1 \le j \le r$$

and

$$\int_{\mathbb{R}} x^{n_j} H_{\vec{n}}(x) e^{-x^2 + c_j(x)} dx = g^{n_j}(c_j) = 2^{-|\vec{n}|} \sqrt{\pi} n_j! \prod_{i \ne j} (c_j - c_i)^{n_i} e^{c_j^2/4}. \tag{2.32}$$

Equation (2.32) shows that H_n is a multiple Hermite polynomial. If we use (2.32) and by using the Section 1.5, we compute the coefficients of the recurrence relations as follows

$$a_{\vec{n},j} = \frac{\int_{\mathbb{R}} x^{n_j} H_{\vec{n}}(x) d\mu_j(x)}{\int_{\mathbb{R}} x^{n_j-1} H_{\vec{n}-\vec{e}_j}(x) d\mu_j(x)} \tag{2.33}$$

$$= \frac{\int_{\mathbb{R}} x^{n_j} H_{\vec{n}}(x) e^{-x^2+c_j} dx}{\int_{\mathbb{R}} x^{n_j-1} H_{\vec{n}-\vec{e}_j}(x) e^{-x^2+c_j} dx} \tag{2.34}$$

$$= \frac{g^{(n_j)}(c_j)}{g^{(n_j-1)}(c_j)}. \tag{2.35}$$

Thus some computation gives

$$a_{\vec{n},j} = \frac{2^{-|\vec{n}|}\sqrt{\pi}n_j! \prod_{i\neq j}(c_j - c_i)e^{c_j^2/4}}{2^{-|\vec{n}-\vec{e_j}|}\sqrt{\pi}(n_j-1)! \prod_{i\neq j}(c_j - c_i)e^{c_j^2/4}} = \frac{n_j}{2}, \qquad 1 \leq j \leq r. \tag{2.36}$$

And by comparing the coefficient of $x^{|\vec{n}|}$ we find

$$b_{\vec{n},j} = \frac{c_j}{2} \tag{2.37}$$

Using (2.36) and (2.37) in (1.9), we conclude that multiple Hermite polynomials satisfy a system of recurrence relations connecting the nearest neighbors given by

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e_j}}(x) + \frac{c_j}{2}H_{\vec{n}}(x) + \frac{1}{2}\sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e_j}}(x), \quad 1 \leq j \leq r. \tag{2.38}$$

2.7 Integral representations

2.7.1 Multiple Hermite polynomials of type II

Theorem 2.7.1. Multiple Hermite polynomials of type II have the integral representation

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty} e^{(s-x)^2}\prod_{l=1}^{r}(s - \frac{c_l}{2})^{n_l} ds. \tag{2.39}$$

Proof. Denote the right hand side of (2.39) by $H(x)$.
We are going to show that H is the multiple Hermite polynomial.After changing variables

$$s - x = t$$

one has

$$H(x) = \frac{1}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty} e^{t^2}\prod_{l=1}^{r}(t + x - \frac{c_l}{2})^{n_l} dt. \tag{2.40}$$

Equation (2.39) shows that H is a polynomial of degree $|\vec{n}| = n_1 + n_2 + ... + n_r$ with leading coefficient

$$\frac{1}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty} e^{t^2} dt = 1.$$

This means that H is a monic polynomial.
We use (2.39) to compute for $j = 1, 2, ..., r$ and $k = 0, ..., n_j - 1$

$$\int_{-\infty}^{\infty} H(x)x^k e^{-x^2+c_jx}dx = \frac{1}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} e^{-x^2+c_jx+t^2}x^k\prod_{l=1}^{r}(t + x - \frac{c_l}{2})^{n_l} dxdt \tag{2.41}$$

Remark that
$-x^2 + c_jx + t^2 = -[(x - \frac{c_j}{2})^2 - \frac{c_j^2}{4}] + t^2 = [t^2 - (x - \frac{c_j}{2})^2] + \frac{c_j^2}{4}$.
Then (2.41) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} H(x)x^k e^{-x^2+c_jx}dx &= \frac{1}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} e^{[t^2-(x-\frac{c_j}{2})^2]+\frac{c_j^2}{4}} x^k \prod_{l=1}^{r}(t + x - \frac{c_l}{2})^{n_l} dxdt \\ &= \frac{e^{\frac{c_j^2}{4}}}{\sqrt{\pi}i}\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} e^{t^2-(x-\frac{c_j}{2})^2} x^k \prod_{l=1}^{r}(t + x - \frac{c_l}{2})^{n_l} dxdt.\end{aligned}$$

Switching to polar coordinates

$$\begin{cases} x - \frac{c_j}{2} = r\cos\theta \\ t = ir\sin\theta, \quad r > 0, \;\; 0 \leq \theta \leq 2\pi. \end{cases}$$

The Jacobian of this transformation is $J(r, \theta) = irdrd\theta$ and then,

$$\begin{aligned}\int_{-\infty}^{\infty} H(x)x^k e^{-x^2+c_jx}dx &= \frac{e^{\frac{c_j^2}{4}}}{\sqrt{\pi}i}\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2}(\frac{c_j}{2} + r\cos\theta)^k \prod_{l=1}^{r}(re^{i\theta} + \frac{c_j}{2} - \frac{c_l}{2})^{n_l} drd\theta \\ &= \frac{e^{\frac{c_j^2}{4}}}{\sqrt{\pi}i}\int_{0}^{\infty} r^{n_j+1}e^{-r^2}[\int_{0}^{2\pi}(\frac{c_j}{2} + r\cos\theta)^k e^{in_j}\prod_{l\neq j}(re^{i\theta} + \frac{c_j}{2} - \frac{c_l}{2})^{n_l} d\theta]dr\end{aligned}$$

where in the product, one separated the terms where $l = j$ and where $l \neq j$. The θ-integral vanishes for $k = 0, ..., n_j - 1$, since the integrand can be written as a linear combination of $e^{ip\theta}$ with ineger $p \geq n_j - k$.
Hence H is the multiple Hermite polynomials of type II and then (2.39) follows. □

We remark that evaluating the last equality for $j = n_k$ we find that the θ-integral is

$$2\pi(\frac{r}{2})\prod_{l\neq k}(a_k - a_l)^{n_l}$$

and

$$\begin{aligned}h_{\vec{n}}^{(k)} &= \int_{-\infty}^{\infty} H_{\vec{n}}(x)x^{n_k}e^{-x^2+c_jx}dx \\ &= \sqrt{2\pi}(n_k)!e^{c_k^2/4}\prod_{l\neq k}(\frac{c_k}{2} - \frac{c_l}{2})^{n_l}. \end{aligned} \tag{2.42}$$

2.7.2 Multiple Hermite polynomials of type I

Theorem 2.7.2. Multiple Hermite polynomials of type I have the integral representation

$$A_{\vec{n},j}(x)e^{-x^2+c_j} = \frac{1}{\sqrt{\pi}2\pi i}\oint_{\Gamma_j} e^{-(t-x)^2}\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}dt \tag{2.43}$$

where Γ_j is a closed contour encircling $\frac{c_j}{2}$ once in the positive direction but not enclosing any of the other $\frac{c_l}{2}, l \neq j$.
In addition, the type I function

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x)e^{-x^2+c_j}$$

has the integral representation

$$Q_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}2\pi i}\oint_{\Gamma} e^{-(t-x)^2}\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}dt \tag{2.44}$$

where Γ is now a closed contour encircling all $\frac{c_l}{2}, (1 \leq l \leq r)$ once in positive direction.

Proof. We first use the residue theorem

$$\text{res}f(a) = \frac{1}{(n-1)!}\lim_{z\longmapsto a}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$

where $z = a$ is a pole with multiplicity n. For our case, $t = \frac{c_j}{2}$ is a pole with multiplicity n_k.Then

$$Q_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}2\pi i} = \frac{1}{(n_j-1)!}\lim_{t\to\frac{c_j}{2}}\frac{d^{n_j-1}}{dt^{n_j-1}}[e^{-(t-x)^2}\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}(t-\frac{c_l}{2})^{n_j}].$$

In this product, one can separate the terms where $n_l = n_k$ and also where $n_l \neq n_k$, then one has

$$Q_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}2\pi i} = \frac{1}{(n_j-1)!}\frac{d^{n_j-1}}{dt^{n_j-1}}[e^{-(t-x)^2}\prod_{l\neq j}(t-\frac{c_l}{2})^{-n_l}]|_{t=\frac{c_j}{2}}. \tag{2.45}$$

We see that (2.45) has the form $A_j(x)e^{-x^2+c_jx}$ where A_j is a polynomial of degree n_j-1.
Define the linear form

$$Q(x) = \sum_{j=1}^{r} A_j(x)e^{-x^2+c_jx} = \frac{1}{\sqrt{\pi}2\pi i}\oint_{\Gamma} e^{-(t-x)^2}\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}dt.$$

where Γ encloses all points $a_k, k = 1, ..., r$, once in the positive direction.
We are going to show that $Q = Q_{\vec{n}}$ and $A_j = A_{\vec{n},j}$ and the theorem will be proved. Then

$$\int_{-\infty}^{\infty} x^k Q(x)dx = \frac{1}{2\pi}\oint_{\Gamma}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^k e^{-(t-x)^2}dx\prod_{l=1}^{r}(t-\frac{c_l}{2})^{-n_l}dt.$$

If we put $y = t - x$, we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^k e^{-(t-x)^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (y+t)^k e^{-y^2} dy = \Pi_k(t).$$

Note that $\Pi_k(t)$ is a polynomial of degree k in the variable t. One has

$$\int_{-\infty}^{\infty} x^k Q(x) dx = \frac{1}{2\pi} \oint_{\Gamma} \Pi_k(t) \prod_{l=1}^{r} (t - \frac{c_l}{2})^{-n_l} dt.$$

Deforming the contour Γ to infinity and then using the fact that the integrand is $s^{k-|\vec{n}|} + O(s^{k-|\vec{n}|-1})$ and as $s \to \infty$,we find that

$$\int_{-\infty}^{\infty} x^k Q(x) dx = 0, \qquad k = 1, ..., |\vec{n}| - 2$$

and $s^{k-|\vec{n}|} + O(s^{k-|\vec{n}|-1})$ and as $s \to \infty$,we find that

$$\int_{-\infty}^{\infty} x^k Q(x) dx = 1, \qquad k = 1, ..., |\vec{n}| - 1$$

Therefore $Q = Q_{\vec{n}}$ and $A_j = A_{\vec{n},j}$.
We conclude that $Q_{\vec{n}}$ are the multiple Hermite polynomials of type I and then (2.43) and (2.45) follow. □

2.7.3 Multiple Hermite kernel

The Brézin-Hikami kernel is defined by

$$K(x,y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{(s-x)^2-(t-y)^2} \prod_{k=1}^{r} (\frac{s - \frac{c_k}{2}}{t - \frac{c_k}{2}})^{n_k} \frac{1}{s-t} \tag{2.46}$$

where Γ is a closed contour encircling the points $a_1, ..., a_r$ once in the positive direction, and the path from $-i\infty$ to $i\infty$ does not intersect Γ.
This kernel agrees with the multiple Hermite kernel defined by the Christoffel-Darboux formula for the case of multiple Hermite polynomials:

$$(x-y)K(x,y) = H_{\vec{n}}(x)Q_{\vec{n}}(y) = -\sum_{k=1}^{r} a_{\vec{n},k} H_{\vec{n}-\vec{e}_k}(x) Q_{\vec{n}+\vec{e}_k}(y) \tag{2.47}$$

where $a_{\vec{n},k}$ is given by (2.36) where $\vec{e}_k$ is the kth standard basis vector in $\mathbb{R}^r$, $H_{\vec{n}}$ and $Q_{\vec{n}}$ are respectively the type II and type I multiple Hermite polynomials.

We can then compute $\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}$ for the kernel (2.46) in two ways:
First way

$$\begin{aligned}\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y} &= \frac{1}{(\sqrt{2}\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{k=1}^{r}(\frac{s-a_k}{t-a_k})^{n_k}\frac{-s+x+t-y}{s-t}\\ &= (x-y)K(x,y)-\frac{1}{(\sqrt{2}\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{k=1}^{r}(\frac{s-\frac{c_k}{2}}{t-\frac{c_k}{2}})^{n_k}.\end{aligned}$$

By (2.40) and (2.45), the double integral is a product of $H_{\vec{n}}(x)$ and $Q_{\vec{n}}(x)$.Then

$$\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}=(x-y)K(x,y)-H_{\vec{n}}(x)Q_{\vec{n}}(x). \tag{2.48}$$

For the second way, we first evaluate $\frac{\partial K}{\partial x}$ by noting that $\partial_x e^{(s-x)^2}=-\partial_s e^{(s-x)^2}$ and integrating by part the $s-$integral

$$\frac{\partial K}{\partial x}=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt\partial_s e^{(s-x)^2-(t-y)^2}\prod_{k=1}^{r}(\frac{s-\frac{c_k}{2}}{t-\frac{c_k}{2}})^{n_k}\frac{1}{s-t}.$$

Then

$$\frac{\partial K}{\partial x}=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{k=1}^{r}(\frac{s-\frac{c_k}{2}}{t-\frac{c_k}{2}})^{n_k}\frac{1}{s-t}\{\sum_{k=1}^{r}\frac{n_k}{s-\frac{c_k}{2}}-\frac{1}{s-t}\}. \tag{2.49}$$

Similarly, we use $\partial_y e^{(s-x)^2}=-\partial_t e^{(s-x)^2}$ and apply integration by part to the t-integral to obtain

$$\frac{\partial K}{\partial y}=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{k=1}^{r}(\frac{s-\frac{c_k}{2}}{t-\frac{c_k}{2}})^{n_k}\frac{1}{s-t}\{-\sum_{k=1}^{r}\frac{n_k}{t-\frac{c_k}{2}}-\frac{1}{s-t}\}. \tag{2.50}$$

We add (2.49) and (2.50) and obtain

$$\begin{aligned}\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y} &= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{j=1}^{r}(\frac{s-\frac{c_j}{2}}{t-\frac{c_j}{2}})^{n_j}\frac{1}{s-t}\sum_{k=1}^{r}(\frac{n_k}{s-\frac{c_k}{2}}-\frac{n_k}{t-\frac{c_k}{2}})\\ &= -\sum_{k=1}^{r}n_k\frac{1}{(2\pi)^2}\int_{-i\infty}^{i\infty} ds \oint_\Gamma dt e^{(s-x)^2-(t-y)^2}\prod_{j\neq k}(\frac{s-\frac{c_j}{2}}{t-\frac{c_j}{2}})^{n_j}\frac{(s-\frac{c_k}{2})^{n_k-1}}{(t-\frac{c_k}{2})^{n_k+1}}.\end{aligned}$$

For every k, the last double integral factors into a product of two single integrales,which by (2.40) and (2.45) are given in terms of multiple Hermite polynomials.
It leads to

$$\frac{\partial K}{\partial x}+\frac{\partial K}{\partial y}=-\sum_{k=1}^{r}n_k H_{\vec{n}-\vec{e_k}}(x)Q_{\vec{n}+\vec{e_k}}(y)$$

which agrees with (2.47) since

$$\frac{n_k}{2}=a_{\vec{n},k}$$

because of (2.42).

Chapter 3

Generating function for the multiple Hermite polynomials

The main reference used in this chapter is [2].

A generating function for multiple Hermite polynomials has been obtained by Lee in [2, Theorem 2.1] for $t_i \to t_i/2$ and his proof used the Rodrigues formula and the Cauchy integral formula. In this chapter, we give two more proofs, one using the function $g_r(t)$ defined in (2.31) and the other using the explicit expression of the multiple Hermite polynomials defined in (2.21).
Some of Lee's results in [2] contain errors, for example the Theorem 2.4 which talks about the interesting recurrence relations, in this chapter we give the correct result in the Theorem 3.0.3.

Definition 3.0.1. The generating function $G(x, t_1, t_1, ..., t_1)$ for multiple orthogonal polynomials $P_{\vec{n}}(x)$ is defined by

$$G(x, t_1, t_2, ..., t_r) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_r=0}^{\infty} \frac{P_{\vec{n}}(x)}{\vec{n}!} t_1^{n_1} t_2^{n_2} ... t_r^{n_r}, \tag{3.1}$$

where $\vec{n}! = n_1!n_2!...n_r!$ is the multi-index factorial function.
We also use by $G(x; \mathbf{t}) := G(x, t_1, t_2, ..., t_r)$,

$$\sum_{\vec{n}=0}^{\infty} := \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} ... \sum_{n_r=0}^{\infty}$$

and $\mathbf{t}^{\vec{n}} := t_1^{n_1} t_2^{n_2} ... t_r^{n_r}$ so that the equation (3.1) is denoted by

$$G(x; \mathbf{t}) = \sum_{\vec{n}=0}^{\infty} \frac{P_{\vec{n}}(x)}{\vec{n}!} \mathbf{t}^{\vec{n}}. \tag{3.2}$$

Theorem 3.0.2. Let $H_{\vec{n}}(x)$ be the multiple Hermite polynomials defined by the equation (2.21).Then the generating function is given by

$$G(x;\mathbf{t}) = \exp\Big(x\sum_{j=1}^{r} t_j - \frac{1}{4}(\sum_{j=1}^{r} t_j)^2 - \frac{1}{2}\sum_{j=1}^{r} c_j t_j\Big). \tag{3.3}$$

In particular, if $r = 2$ then

$$G(x;t_1,t_2) = \exp\Big[x(t_1+t_2) - \frac{1}{4}(t_1^2 + 2t_1t_2 + t_2^2) - \frac{1}{2}(c_1t_1 + c_2t_2)\Big]. \tag{3.4}$$

Proof. **First method**

We consider the function $g_r(t)$ defined in (2.31) as

$$g_r(t) = \int_{\mathbb{R}} e^{tx} H_{\vec{n}}(x) e^{-x^2} dx = 2^{-|\vec{n}|}\sqrt{\pi}(t-c_1)^{n_1}...(t-c_r)^{n_r} e^{t^2/4}, \tag{3.5}$$

Then one has

$$\begin{aligned} \int_{\mathbb{R}} e^{tx} \frac{H_{\vec{n}}(x) t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} e^{-x^2} dx &= 2^{|\vec{n}|}\sqrt{\pi} e^{t^2/4} \frac{[(t_1-c_1)t_1]^{n_1}}{n_1!} \cdots \frac{[(t_r-c_r)t_r]^{n_r}}{n_r!} \\ \int_{\mathbb{R}} e^{tx} \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \frac{H_{\vec{n}}(x) t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} e^{-x^2} dx &= \sqrt{\pi} e^{t^2/4} \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \frac{[(t_1-c_1)\frac{t_1}{2}]^{n_1}}{n_1!} \cdots \frac{[(t_r-c_r)\frac{t_r}{2}]^{n_r}}{n_r!}, \end{aligned} \tag{3.6}$$

so that

$$\begin{aligned} \int_{\mathbb{R}} e^{tx} G(x;\mathbf{t}) e^{-x^2} dx &= \sqrt{\pi} e^{t^2/4} \exp((t_1-c_1)\frac{t_1}{2}) \ldots \exp((t_r-c_r)\frac{t_r}{2}) \\ &= \sqrt{\pi} e^{t^2/4} \exp(\frac{1}{2}\sum_{j=1}^{r}(t_j-c_j)t_j) \\ &= \sqrt{\pi} e^{t^2/4} e^{tA} \exp(-\frac{1}{2}\sum_{j=1}^{r} c_j t_j) \end{aligned} \tag{3.7}$$

where

$$A = \frac{1}{2}\sum_{j=1}^{r} t_j.$$

For $\vec{n} = (0,\ldots,0)$, (3.5) becomes

$$\int_{\mathbb{R}} e^{tx} e^{-x^2} dx = \sqrt{\pi} e^{t^2/4} e^{tA}. \tag{3.8}$$

Multiplying the both sides hands of (3.8) by e^{tA}, one has

$$\int_{\mathbb{R}} e^{-x^2} e^{t(x+A)} dx = \sqrt{\pi} e^{t^2/4} e^{tA}.$$

After changing variable $x + A = y$ and $y \to x$, one has

$$\int_{\mathbb{R}} e^{-(x-A)^2} e^{tx} dx = \sqrt{\pi} e^{t^2/4} e^{tA}. \tag{3.9}$$

Comparing (3.7) and (3.9)

$$\begin{aligned} G(x;\mathbf{t}) e^{-x^2} \exp(\frac{1}{2}\sum_{j=1}^{r} c_j t_j) &= e^{-(x-A)^2} \\ G(x;\mathbf{t}) e^{-x^2} &= \exp\left[-(x - \frac{1}{2}\sum_{j=1}^{r} t_j)^2\right] \exp(-\frac{1}{2}\sum_{j=1}^{r} c_j t_j) \\ &= \exp[-x^2 + x\sum_{j=1}^{r} t_j - \frac{1}{4}(\sum_{j=1}^{r} t_j)^2 - \frac{1}{2}\sum_{j=1}^{r} c_j t_j] \end{aligned} \tag{3.10}$$

Finally, the generating function of the multiple Hermite polynomials is given by

$$G(x;\mathbf{t}) = \exp\Big(x\sum_{j=1}^{r} t_j - \frac{1}{4}(\sum_{j=1}^{r} t_j)^2 - \frac{1}{2}\sum_{j=1}^{r} c_j t_j\Big). \tag{3.11}$$

Second method

Now, we use the explicit expression of the multiple Hermite polynomials defined in (2.21). The expression (2.21) can be rewritten as

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{H_{\vec{n}}(x)}{n_1! \ldots n_r!} t_1^{n_1} \ldots t_r^{n_r} \\ &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \frac{\binom{n_1}{k_1}}{n_1!} \cdots \frac{\binom{n_r}{k_r}}{n_r!} t_1^{n_1} c_1^{n_1-k_1} \ldots t_r^{n_r} c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x) \\ &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \frac{(t_1 c_1)^{n_1} c_1^{-k_1}}{k_1!(n_1-k_1)!} \cdots \frac{(t_r c_r)^{n_r} c_r^{-k_r}}{k_r!(n_r-k_r)!} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x) \\ &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{c_1^{-k_1} \ldots c_r^{-k_r}}{k_1! \ldots k_r!} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x) \sum_{n_1=k_1}^{\infty} \cdots \sum_{n_r=k_r}^{\infty} (-1)^{|\vec{n}|} \frac{(\frac{t_1 c_1}{2})^{n_1}}{(n_1-k_1)!} \cdots \frac{(\frac{t_r c_r}{2})^{n_r}}{(n_r-k_r)!}. \end{aligned}$$

The last equality results from

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty}$$

If we put in the last sums $n_1 - k_1 = j_1, \ldots, n_r - k_r = j_r$ and $j_1 \to k_1, \ldots, j_r \to k_r$, one has

$$
\begin{aligned}
&\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{H_{\vec{n}}(x)}{n_1! \ldots n_r!} t_1^{n_1} \ldots t_r^{n_r} \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{c_1^{-k_1} \ldots c_r^{-k_r}}{k_1! \ldots k_r!} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x) \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} (-1)^{2k_1} \frac{(\frac{t_1 c_1}{2})^{2k_1}}{(k_1)!} \ldots (-1)^{2k_r} \frac{(\frac{t_r c_r}{2})^{2k_r}}{(k_r)!} \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{(-1)^{2|\vec{k}|}}{k_1! \ldots k_r!} H_{|\vec{k}|}(x) \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \frac{\frac{t_1}{2}}{(k_1)!} (\frac{-t_1 c_1}{2})^{k_1} \ldots \frac{\frac{t_r}{2}}{(k_r)!} (\frac{-t_r c_r}{2})^{k_r} \\
&= \sum_{k=0}^{\infty} \cdots \sum_{k=|\vec{k}|} \frac{H_k(x)}{k_1! \ldots k_r!} \frac{t_1^{k_1 \ldots t_r^{k_r}}}{2^k} \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \\
&= \sum_{k=0}^{\infty} \frac{H_k(x)}{2^k k!} \sum_{k=|\vec{k}|} \frac{k!}{k_1 \ldots k_r} t_1^{k_1} \ldots t_r^{k_r} \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \\
&= \sum_{k=0}^{\infty} \frac{H_k(x)}{2^k k!} \sum_{k=|\vec{k}|} \binom{k}{k_1, \ldots, k_r} t_1^{k_1} \ldots t_r^{k_r} \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big).
\end{aligned}
$$

We use the multinomial theorem

$$
\sum_{|\vec{k}|=n} \binom{n}{k_1, \ldots, k_r} t_1^{k_1} \ldots t_r^{k_r} = (t_1 + \ldots + t_r)^n
$$

and obtain

$$
\begin{aligned}
G(x; \mathbf{t}) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{H_{\vec{n}}(x)}{n_1! \ldots n_r!} t_1^{n_1}, \ldots, t_r^{n_r} \\
&= \sum_{k=0}^{\infty} \frac{H_k(x)}{2^k k!} (t_1 + \ldots + t_r)^k \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \\
&= \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} (\frac{1}{2} \sum_{j=}^{r} t_j)^k \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \\
&= G(x; \frac{1}{2} \sum_{j=1}^{r} t_j) \exp\Big(- \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \\
&= \exp\Big(2x (\frac{1}{2} \sum_{j=1}^{r} t_j) - \frac{1}{4} (\sum_{j=1}^{r} t_j)^2 - \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big),
\end{aligned}
$$

Finally, we have

$$
G(x; \mathbf{t}) = \exp\Big(x \sum_{j=1}^{r} t_j - \frac{1}{4} (\sum_{j=1}^{r} t_j)^2 - \frac{1}{2} \sum_{j=1}^{r} c_j t_j \Big) \tag{3.12}
$$

and (3.4) is its special case for $r = 2$. $\square$

By this theorem, we obtain an interesting recurrence relation.

Theorem 3.0.3. For multiple Hermite polynomials, we have the following interesting recurrence relation

$$H_{\vec{n}+\vec{e}_i}(x) = (x - \frac{c_i}{2})H_{\vec{n}}(x) - \frac{1}{2}\sum_{j=1}^{r} n_j H_{\vec{n}+\vec{e}_j}(x), \quad 1 \leq i \leq r. \tag{3.13}$$

Proof. The derivative of the generating function G in (3.11) with respect to x is given by

$$\partial_x G(x;\mathbf{t}) = \sum_{j=1}^{r} t_j G(x;\mathbf{t}) = \sum_{\vec{n}=0}^{\infty}\sum_{j=1}^{r} \frac{t_j H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!}. \tag{3.14}$$

By the definition of the generating function in general and the the raising operator of the multiple Hermite polynomials, one has

$$\partial_x G(x;\mathbf{t}) = \sum_{\vec{n}=0}^{\infty} \frac{H'_{\vec{n}}(x)}{\vec{n}!}\mathbf{t}^{\vec{n}} = \sum_{\vec{n}=0}^{\infty}\sum_{j=1}^{r} \frac{n_j H_{\vec{n}-\vec{e}_j}(x)}{\vec{n}!}\mathbf{t}^{\vec{n}}. \tag{3.15}$$

But

$$\frac{n_j}{\vec{n}!} = \frac{n_j}{n_1!\dots n_j!\dots n_r!} = \frac{1}{(\vec{n}-\vec{e}_j)!}$$

Thus

$$\partial_x G(x;\mathbf{t}) = \sum_{\vec{n}=0}^{\infty}\sum_{j=1}^{r} \frac{H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!}\mathbf{t}^{\vec{n}}. \tag{3.16}$$

By comparing (3.14) and (3.16), one has

$$\frac{H'_{\vec{n}}(x)}{\vec{n}!} = \sum_{j=1}^{r} \frac{H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!}$$

It means that

$$H'_{\vec{n}}(x) = \sum_{j=1}^{r} \frac{\vec{n}! H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!}$$

But

$$\frac{\vec{n}!}{(\vec{n}-\vec{e}_j)!} = \frac{n_1!\dots n_j!\dots n_r!}{n_1!\dots(n_j-1)!\dots n_r!} = n_j$$

Then, one has

$$H'_{\vec{n}}(x) = \sum_{j=1}^{r} n_j! H_{\vec{n}-\vec{e}_j}(x) \tag{3.17}$$

On the one hand, the derivative of the generating function G in (3.11) with respect to t_i is given by

$$\partial_{t_i} G(x;\mathbf{t}) = (x - \frac{1}{2}\sum_{j=1}^{r} t_j - \frac{1}{2}c_i)G(x;\mathbf{t}), \qquad 1 \le i \le r. \tag{3.18}$$

On the other hand, one has

$$\begin{aligned}\partial_{t_i} G(x;\mathbf{t}) &= \sum_{n_1=0}^{\infty}\dots\sum_{n_i=0}^{\infty}\dots\sum_{n_r=0}^{\infty}\frac{H_{\vec{n}}(x)n_i t_1^{n_1}\dots t_i^{n_i-1}\dots t_r^{n_r}}{n_1!...n_i!...n_r!}\\ &= \sum_{n_1=0}^{\infty}\dots\sum_{n_i=0}^{\infty}\dots\sum_{n_r=0}^{\infty}\frac{H_{\vec{n}}(x)t_1^{n_1}\dots t_i^{n_i-1}\dots t_r^{n_r}}{n_1!...(n_i-1)!...n_r!}.\end{aligned}$$

If we take $n_i \to n_i + 1$, one has

$$\begin{aligned}\partial_{t_i} G(x;\mathbf{t}) &= \sum_{n_1=0}^{\infty}\dots\sum_{n_i=0}^{\infty}\dots\sum_{n_r=0}^{\infty}\frac{H_{\vec{n}+\vec{e}_i}(x)t_1^{n_1}\dots t_i^{n_i}\dots t_r^{n_r}}{n_1!...(n_i)!...n_r!}\\ &= \sum_{\vec{n}=0}^{\infty}\frac{H_{\vec{n}+\vec{e}_i}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!}.\end{aligned} \tag{3.19}$$

Then (3.18) becomes

$$\begin{aligned}\partial_{t_i} G(x;\mathbf{t}) &= (x - \frac{1}{2}\sum_{j=1}^{r} t_j - \frac{1}{2}c_i)\sum_{\vec{n}=0}^{\infty}\frac{H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!}\\ &= x\sum_{\vec{n}=0}^{\infty}\frac{H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!} - \frac{1}{2}\sum_{j=1}^{r} t_j \sum_{\vec{n}=0}^{\infty}\frac{H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!} - \frac{1}{2}\sum_{\vec{n}=0}^{\infty} c_i \frac{H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!}.\end{aligned} \tag{3.20}$$

By (3.15) and (3.18), one has

$$\sum_{j=1}^{r} t_j \sum_{\vec{n}=0}^{\infty}\frac{H_{\vec{n}}(x)\mathbf{t}^{\vec{n}}}{\vec{n}!} = \partial_x G(x;\mathbf{t}) = \sum_{\vec{n}=0}^{\infty}\sum_{j=1}^{r}\frac{H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!}\mathbf{t}^{\vec{n}}.$$

Then (3.20) becomes

$$\begin{aligned}\partial_{t_i} G(x;\mathbf{t}) &= \sum_{\vec{n}=0}^{\infty}\Big(xH_{\vec{n}}(x) - \frac{1}{2}\sum_{j=1}^{r}\vec{n}!\frac{H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!} - \frac{1}{2}c_i H_{\vec{n}}(x)\Big)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}\\ &= \sum_{\vec{n}=0}^{\infty}\Big((x - \frac{1}{2}c_i)H_{\vec{n}}(x) - \frac{1}{2}\sum_{j=1}^{r}\vec{n}!\frac{H_{\vec{n}-\vec{e}_j}(x)}{(\vec{n}-\vec{e}_j)!}\Big)\frac{\mathbf{t}^{\vec{n}}}{\vec{n}!}.\end{aligned} \tag{3.21}$$

By comparing (3.19) and (3.21), one has finally the following interesting recurrence relations

$$H_{\vec{n}+\vec{e}_i}(x) = (x - \frac{c_i}{2})H_{\vec{n}}(x) - \frac{1}{2}\sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x), \quad 1 \le i \le r. \tag{3.22}$$

$\square$

Chapter 4

Riemann-Hilbert problem and multiple orthogonal polynomials

In this Chapter we refer to the references [14], [11], [15], [16], [3], [4] and [9].

Multiple Hermite polynomials are useful for investigating random matrices with external source and especially the Gaussian unitarly ensemble(GUE).In this chapter we show the connection between the two notions.For the case of multiple Laguerre polyonomials we have a relation with the Wishart ensemble of random matrices.

4.1 Random matrices and Gaussian unitary ensemble(GUE)

Let M be a random Hermitian matrix of order n with probability density

$$\frac{1}{Z_n} e^{-\frac{1}{2}\mathrm{Tr}(M^2)} dM$$

Then
$M_{j,j} = N(0,1), \qquad 1 \leq j \leq n$

$\Re(M_{i,j}) = N(0,\frac{1}{2}), \quad \Im(M_{i,j}) = N(0,\frac{1}{2}), \qquad 1 \leq i < j \leq n$

and these matrix elements are all independent.
Then

$$\mathbb{E}\det(M - zI_n) = (-1)^n H_n(z)$$

where H_n is the **probabilistic Hermite polynomial** of order n.

$$\int_{-\infty}^{\infty} x^k H_n(x) e^{-\frac{x^2}{2}} dx = 0, \quad 0 \leq k \leq n-1. \tag{4.1}$$

4.2 Random matrices with external source and multiple orthogonal polynomials

We consider a random Hermitian matrix M with external source A

$$\frac{1}{Z_{|\vec{n}|}} e^{-\mathrm{Tr}(V(M)-AM)} dM \tag{4.2}$$

defined on $|\vec{n}|\times|\vec{n}|$ Hermitian matrices M, where

$$dM = \prod_{i=1}^{|\vec{n}|} dM_{i,i} \prod_{1\le i<j\le|\vec{n}|} d\Re M_{i,j} d\Im M_{i,j}$$

and

$$V : \mathbb{R} \to \mathbb{R}$$

is a function with enough increase at $\pm\infty$ such that the integral

$$Z_{|\vec{n}|} = \int e^{-\mathrm{Tr}(V(M)-AM)} dM$$

converges.
The ensemble (4.2) consists of a general unitary invariant part $V(M)$ and an extra term AM, where A is a fixed Hermitian matrix of order $|\vec{n}|$ called the external source or the external field.
Due to the external source, the ensemble (4.2) is not unitarily invariant.

For the special Gaussian case $V(x) = \dfrac{1}{2}x^2$,we write M in (4.2) as $M = H + A$, where H is a random matrix from the Gaussian unitary ensemble and A is deterministic.

Let then M be a random matrix of order $|\vec{n}|$ and consider a multi-index $\vec{n} = (n_1, n_2, ..., n_r)$ and its length $|\vec{n}| = n_1 + n_2 + ... + n_r$ and A a fixed Hermitian matrix.
The probability density for M is

$$\frac{1}{Z_{|\vec{n}|}} e^{-\mathrm{Tr}(M^2-AM)} dM$$

If A has n_j eigenvalues c_j, $(1 \le i < j)$

$$\mathbb{E}[\det(M - zI_{|\vec{n}|})] = (-1)^{|\vec{n}|} H_{\vec{n}}(z) \tag{4.3}$$

$H_{\vec{n}}$ is a **multiple Hermite polynomial** and satisfies

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k e^{-x^2+c_j x} dx = 0, \quad 0 \le k \le n_j - 1, \quad 1 \le j \le r. \tag{4.4}$$

Zinn-Justin showed in [16] that the eigenvalue correlations of ensemble (4.2) can be expressed in the determinental form

$$R_m(\lambda_1, ..., \lambda_m) = \det(K_{\vec{n}}(\lambda_i, \lambda_j))_{i,j=1,...,n} \tag{4.5}$$

and P.M.Bleher and A.B.J.Kuijlaars proved in [15] that the joint probability density of eigenvalues has the determinental form

$$\frac{1}{\vec{n}!} \det(K_{\vec{n}}(\lambda_i, \lambda_j))_{i,j=1,...,m}$$

for some kernel $K_{\vec{n}}$.
The average characteristic polynomial of ensemble (4.2) defined in (4.3) can characterized by the property that

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k e^{-(V(x)-c_j x)} dx = 0 \tag{4.6}$$

for every eigenvalue c_j of A and for $k = 0, ..., n_j - 1$ and where n_j is the multiplicity of c_j. This means that $H_{\vec{n}}$ is a **multiple Hermite polynomial** of type II with multi-index $\vec{n} = (n_1, ..., n_r)$ when A has r distinct eigenvalues $c_1, ..., c_r$ respectively.

The kernel $K_{\vec{n}}$ has the form

$$K_{\vec{n}}(x, y) = e^{-(V(x)-V(y))} \sum_{k=0}^{|\vec{n}|-1} P_k(x) Q_k(y), \tag{4.7}$$

where the type I Q_k are certain functions (not polynomials in general).

Theorem 4.2.1. The Christoffel-Darboux kernel satisfies

$$\int_{-\infty}^{\infty} K_{\vec{n}}(x, y) K_{\vec{n}}(y, z) d\mu(y) = K_{\vec{n}}(x, z) \tag{4.8}$$

and

$$\int_{-\infty}^{\infty} K_{\vec{n}}(x, x) d\mu(x) = |\vec{n}|. \tag{4.9}$$

Proof. By the definition of the Christoffel-Darboux kernel, one has

$$\int_{-\infty}^{\infty} K_{\vec{n}}(x, y) K_{\vec{n}}(y, z) d\mu(y) = \int_{-\infty}^{\infty} P_k(y) Q_k(y) d\mu(y) e^{-(V(x)-V(z))} \sum_{k=0}^{|\vec{n}|-1} P_k(x) Q_k(z) \tag{4.10}$$

Because of (1.1) and (4.7),(4.8) follows.

To prove (4.9), one has

$$\int_{-\infty}^{\infty} K_{\vec{n}}(x, x) d\mu(x) = \sum_{k=0}^{|\vec{n}|-1} \int_{-\infty}^{\infty} P_k(x) Q_k(x) d\mu(x) \tag{4.11}$$

By (1.1) and the fact that

$$\sum_{k=0}^{m-1} 1 = m$$

then (4.9) follows. □

When $A = 0$ (no external source) the polynomials P_k are usual monic orthogonal polynomials with respect to the weight $e^{-V(x)}$ on $\mathbb{R}$.

In that case, the function Q_k is a multiple of P_k and the kernel (4.7) reduces to the orthogonal polynomial kernel which is familiar in the theory of random matrices.
By the Christoffel-Darboux formula, we then have

$$K_n(x,y) = e^{-V(x)-V(y)} a_n \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x-y} \tag{4.12}$$

where $a_n = \frac{\gamma_{n-1}}{\gamma_n}$ and γ_n is the leading coefficient of the orthonormal polynomial of degree n.

4.3 Large n limit of Gaussian random matrices with external source

4.3.1 Introduction

We consider the random matrix ensemble with external source

$$\mu_{\vec{n}}(dM) = \frac{1}{Z_{|\vec{n}|}} e^{-|\vec{n}|Tr(V(M)-AM)} dM \tag{4.13}$$

defined on $|\vec{n}|\times|\vec{n}|$ Hermitian matrices M.
The Gaussian case $V(M) = \frac{1}{2}M^2$ has been solved by Pastur in [9] and by Brézin-Hikami in [3] and [4] by using spectral methods and contour integration formula for the determinental kernel.

Bleher and Arno B.J.Kuijlaars in [14] used an approach based on a Riemann-Hilbert (RH) problem and it is applicable to general V.This approach is discussed in the the section below for $r = 2$.

We assume that the external source A is a fixed diagonal matrix with n_1 eigenvalues a and n_2 eigenvalues $(-a)$ i.e

$$A = diag(\underbrace{a, ..., a}_{n_1 \ times}, \underbrace{-a, ..., -a}_{n_2 \ times})$$

such that $n_1 + n_2 = |\vec{n}|$.

For any $m \geq 1$, as seen in equation (4.5), the $m-$point correlation functions of eigenvalues of M has the determinental form

$$R_m(\lambda_1, ..., \lambda_m) = \det(K_{\vec{n}}(\lambda_i, \lambda_j))_{i,j=1,...,m}.$$

In [14], it is shown that the kernel $K_{\vec{n}}(x, y)$ can be expressed in terms of the solution of a Riemann-Hilbert problem, which is studied in the following section.

4.3.2 Riemann-Hilbert problem

The Cauchy transform

A typical scalar and additive Riemann-Hilbert(RH) problem is to find a function $f : \mathbb{C} \to \mathbb{C}$ such that

- f is analytic on $\mathbb{C} \setminus \mathbb{R}$
- for $x \in \mathbb{R}$,
$$f_+(x) = f_-(x) + \omega(x) \tag{4.14}$$
- as $z \to \infty$, $f(z) = O(\frac{1}{z})$

where $\omega : \mathbb{R} \to \mathbb{R}$ is a given function which describes the jump that f makes as it crosses the real axis.
Suppose that $\omega \in L_1(\mathbb{R})$ and ω is Hölder continuous on $\mathbb{R}$, that is
$|\omega(x) - \omega(y)| \leq c|x - y|^\alpha$, for all $x, y \in \mathbb{R}$
where $c > 0$ is a constant and $0 < \alpha \leq 1$.

Then the unique solution of the RH problem is given by

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(s)}{s - z} ds \tag{4.15}$$

which is the Cauchy transform or Stieltjes transform of the function ω.

The Fokas-Its-Kitaev boundary value problem

The basic idea of the RH problem approach to orthogonal polynomials is to characterize the orthogonal polynomials corresponding to a weight function ω on the real line via a boundary value problem for matrix valued analytic functions. This was first formulated in a ground-breaking paper of Fokas, Its and Kitaev in 1992.
The RH problem for orthogonal polynomials on the real line with a weight function ω is to find a matrix valued function $Y : \mathbb{C} \to \mathbb{C}^{2\times 2}$ which satisfies the following three conditions:

- Y is analytic in $\mathbb{C} \setminus \mathbb{R}$

- (jump condition) On the real line we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}$$

- (normalization near infinity) Y has the following behavior near infinity

$$Y(z) = \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

The boundary values $Y_+(x)$ and $Y_-(x)$ are defined as

$$Y_\pm(x) = \lim_{\epsilon \to 0^\pm} Y(x \pm i\epsilon).$$

Suppose that $x^j \omega \in L^1(\mathbf{R})$ for every $j \in \mathbb{N}$ and that ω is Hölder continuous on $\mathbb{R}$. Then for $n \geq 1$ the solution of that RH problem for Y is given by

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{\pi i} \int_{\mathbb{R}} \frac{P_n(s)\omega(s)}{s-z} ds \\ -2\pi i \gamma_{n-1}^2 P_{n-1}(z) & -\gamma_{n-1}^2 \int_{\mathbb{R}} \frac{P_{n-1}(s)\omega(s)}{s-z} ds \end{pmatrix}$$

where P_n is the monic orthogonal polynomial of degree n for the weight function ω and γ_{n-1} is the leading coefficient of the orthonormal polynomial p_{n-1}.

Riemann–Hilbert problem for Hermite polynomials

Consider the RH problem for Hermite polynomials where $Y : \mathbb{C} \to \mathbb{C}^{2\times 2}$ is a matrix valued function with the following properties:

- Y is analytic in $\mathbb{C} \setminus \mathbb{R}$
- (jump condition) The boundary values Y_+ and Y_- exist on $\mathbb{R}$ and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}$$

- (normalization near infinity) Y has the following behavior near infinity

$$Y(z) = \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Then

$$Y(z) = \begin{pmatrix} h_n(z) & \frac{1}{\pi i} \int_{\mathbb{R}} \frac{h_n(s)\omega(s)}{s-z} ds \\ -2\pi i \gamma_{n-1}^2 h_{n-1}(z) & -\gamma_{n-1}^2 \int_{\mathbb{R}} \frac{h_{n-1}(s)\omega(s)}{s-z} ds \end{pmatrix}$$

where $h_n = 2^{-n} H_n$ are the monic Hermite polynomials.

Riemann-Hilbert problem for multiple orthogonal polynomials

We relate the Christoffel-Darboux formula to a RH problem.

The RH problem for $r = 2$ consists of finding

$$Y : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}^{3\times 3}$$

such that

- Y is analytic on $\mathbb{C} \setminus \mathbb{R}$.
  A matrix function Y is analytic in z if each of its components is an analytic function of z.

- the boundary values

$$Y_{\pm}(x) = \lim_{\epsilon \to 0^{\pm}} Y(x \pm i\epsilon) \text{ exist and for } x \in \mathbb{R},$$

  we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega_1(x) & \omega_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R} \tag{4.16}$$

  where

$$\omega_1(x) = e^{-|\vec{n}|(V(x)-ax)} \quad \text{and} \quad \omega_2(x) = e^{-|\vec{n}|(V(x)+ax)} \tag{4.17}$$

- as $z \to \infty$, we have the asymptotic behavior

$$Y(z) = \left(I + O(\tfrac{1}{z})\right) \begin{pmatrix} z^{|\vec{n}|} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix} \tag{4.18}$$

  where I is the 3×3 identity matrix

- In addition one may need some condition near the endpoints of the supports of measures $\mu_1, ..., \mu_2$

Denote

$$\begin{aligned} P_{\vec{n}} &= P_{(n_1,n_2)}, \\ P_{\vec{n}-\vec{e}_1} &= P_{(n_1-1,n_2)}, \\ P_{\vec{n}-\vec{e}_2} &= P_{(n_1,n_2-1)}. \end{aligned}$$

The solution of RH problem exists when the mutli-index $\vec{n} = (n_1, n_2)$ is normal and it contains the type II multiple orthogonal polynomial $P_{\vec{n}}$ and its neighbors from below

$$Y(z) = \begin{pmatrix} P_{\vec{n}}(z) & \frac{1}{2\pi i}\int \frac{P_{\vec{n}}(x)\omega_1(x)}{x-z}dx & \frac{1}{2\pi i}\int \frac{P_{\vec{n}}(x)\omega_2(x)}{x-z}dx \\ -2\pi i\gamma_1 P_{\vec{n}-\vec{e_1}}(z) & -\gamma_1 \int \frac{P_{\vec{n}-\vec{e_1}}(x)\omega_1(x)}{x-z}dx & -\gamma_1 \int \frac{P_{\vec{n}-\vec{e_1}}(x)\omega_2(x)}{x-z}dx \\ -2\pi i\gamma_2 P_{\vec{n}-\vec{e_2}}(z) & -\gamma_2 \int \frac{P_{\vec{n}-\vec{e_2}}(x)\omega_1(x)}{x-z}dx & -\gamma_2 \int \frac{P_{\vec{n}-\vec{e_2}}(x)\omega_2(x)}{x-z}dx \end{pmatrix} \tag{4.19}$$

where

$$\frac{1}{\gamma_j} = \frac{1}{\gamma_j(\vec{n})} = \int x^{n_j-1} P_{\vec{n}-\vec{e_j}}(x)\omega_j(x)dx, \quad 1 \le j \le 2$$

is the leading coefficient of $P_{\vec{n}}$.

There is a similar of RH problem for the type I multiple orthogonal polynomials:

The RH problem of the type I multiple orthogonal polynomials consists of finding

$$X : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}^{3\times 3}$$

such that

- X is analytic on $\mathbb{C} \setminus \mathbb{R}$
- the boundary values $X_\pm(x) = \lim_{\epsilon \mapsto 0^+} X(x \pm i\epsilon)$ exist and for $x \in \mathbb{R}$, we have

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -\omega_1(x) & 1 & 0 \\ -\omega_2(x) & 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \tag{4.20}$$

  where

$$\omega_1(x) = e^{-|\vec{n}|(V(x)-ax)} \quad \text{and} \quad \omega_2(x) = e^{-|\vec{n}|(V(x)+ax)} \tag{4.21}$$

- as $z \to \infty$, we have the asymptotic behavior

$$X(z) = \left(I + O(\tfrac{1}{z})\right) \begin{pmatrix} z^{-|\vec{n}|} & 0 & 0 \\ 0 & z^{n_1} & 0 \\ 0 & 0 & z^{n_2} \end{pmatrix} \tag{4.22}$$

  where I is the 3×3 identity matrix
- In addition one may need some condition near the endpoints of the supports of measures $\mu_1, ..., \mu_2$.

Denote

$$\begin{aligned} Q_{\vec{n}} &= Q_{(n_1,n_2)}, \\ Q_{\vec{n}+\vec{e}_1} &= Q_{(n_1+1,n_2)}, \\ Q_{\vec{n}+\vec{e}_2} &= Q_{(n_1,n_2+1)}. \end{aligned}$$

The solution of RH problem is in terms of the type I multiple orthogonal polynomial $A_{\vec{n},j}(1 \le j \le 2)$, the functions $Q_{\vec{n}}(x) = \sum_{j=1}^{2} A_{\vec{n},j}(x)\omega_j(x)$ and all the neighbors from above

$$X(z) = \begin{pmatrix} \int \frac{Q_{\vec{n}}(x)}{x-z}dx & 2\pi i A_{\vec{n},1}(z) & 2\pi i A_{\vec{n},1}(z) \\ \frac{c_1}{2\pi i}\int \frac{Q_{\vec{n}+\vec{e_1}}(x)}{x-z}dx & c_1 A_{\vec{n}+\vec{e_1},1}(z) & c_2 A_{\vec{n}+\vec{e_1},2}(z) \\ \frac{c_2}{2\pi i}\int \frac{Q_{\vec{n}+\vec{e_2}}(x)}{x-z}dx & c_2 A_{\vec{n}+\vec{e_2},1}(z) & c_2 A_{\vec{n}+\vec{e_2},2}(z) \end{pmatrix} \tag{4.23}$$

where $\frac{1}{c_j} = \frac{1}{c_j(\vec{n})}$ is the leading coefficient of $A_{\vec{n}+\vec{e_j},j}$.

Remark 4.3.1. The two RH problems are related as follows:

$$X(x) = Y^{-T}(x),$$

where M^{-T} is the transpose of the inverse of the matrix M.

The multiple kernel for $r = 2$ is

$$K_{(n_1,n_2)}(x,y) = \sum_{k=0}^{n_1+n_2-1} P_{n_k}(x)Q_{n_k+1}(y).$$

Denote by

$$\begin{aligned} n_0 &= (0,0), \\ n_{n_1+n_2} &= (n_1,n_2) \text{ and} \\ n_k+1 &= n_k+e_j, \quad j \in \{1,2\}. \end{aligned}$$

The functions $Q_{\vec{n}}$ are defined for $r = 2$ by

$$Q_{\vec{n}}(x) = A_{\vec{n},1}(x)\omega_1(x) + A_{\vec{n},2}(x)\omega_2(x)$$

where $A_{\vec{n},j}$ is a polynomial of degree at most $n_j - 1$ ($1 \le j \le 2$).
For $r = 1$, one has the classical Christoffel-Darboux formula of the form

$$K_n(x,y) = \sum_{k=0}^{n-1} P_k(x)P_k(y) = a_n \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y}$$

where $a_n = \frac{\gamma_{n-1}}{\gamma_n}$ and γ_n is the leading coefficient of the orthonormal polynomial of degree n.
For the multiple orthogonal polynomials, the Christoffel-Darboux formula is given for $r = 2$ by

$$K_{(n_1,n_2)}(x,y) = \sum_{k=0}^{n_1+n_2-1} P_{n_k}(x)Q_{n_k+1}(y) = \frac{P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}_j-\vec{e}_j}(x)Q_{\vec{n}_j+\vec{e}_j}(y)}{x-y}$$

with $a_{\vec{n},j} = \frac{c_j}{2}$.

Then, the compact form for the kernel in terms of the solution of the RH problem in theory of the random matrices is given by [14]

$$\begin{aligned} K_{\vec{n}}(x,y) &= e^{-\frac{1}{2}|\vec{n}|(V(x)+V(y)}\frac{e^{|\vec{n}|ay}[Y^{-1}(y)Y(x)]_{21}+e^{-|\vec{n}|ay[Y^{-1}(y)Y(x)]_{31}}}{2\pi i(x-y)} \\ &= \frac{e^{-\frac{1}{2}|\vec{n}|(V(x)+V(y))}}{2\pi i(x-y)}\begin{pmatrix}0 & e^{|\vec{n}|ay} & e^{-|\vec{n}|ay}\end{pmatrix} Y^{-1}(y)Y(x)\begin{pmatrix}1\\0\\0\end{pmatrix}. \end{aligned} \tag{4.24}$$

Equations (4.19) and (4.23) show that the RH problem gives a unique solution in terms of multiple orthogonal polynomials.

Let us mention that the (1,1) entry in (4.19) satisfies

$$Y_{1,1}(z) = P_{\vec{n}}(z) = \mathbb{E}[\det(zI - M)] \tag{4.25}$$

where $\mathbb{E}$ denotes the expectation with respect to measure $(1,1)$.So it is the average characteristic polyonomial of a random matrix ensemble.

Bleher and Kuijlaars in [14] analysed the RH problem as $|\vec{n}| \to \infty$ by using the method of steepest descent/stationary phase of Deift and Zhou [12]. They focused on the Gaussian case : $V(x) = \frac{1}{2}x^2$.

The approach used in [14] based on RH problem proves simultaneously the asymptotic behavior of the $(1,1)$ entry of Y which by (4.25) is equal to the average characteristic polynomial and this polynomial is called **multiple Hermite polynomial** for the Gaussian case.

4.3.3 Recurrence relations and differential equation

We discuss here the recurrence relations and differential equations that are satisfied by the solution of the RH problem (4.20)-(4.18) in case of $V(x) = \frac{1}{2}x^2$.

For the recurrence relations, we need to separate the indices n_1 and n_2 in the asymptotic behavior (4.18) from the exponent $|\vec{n}|$ in the weight functions ω_1 and ω_2 defined by (4.17):

$$\omega_1(x) = e^{-|\vec{n}|(\frac{1}{2}x^2 - ax)} \quad \text{and} \quad \omega_2(x) = e^{-|\vec{n}|(\frac{1}{2}x^2 + ax)} \tag{4.26}$$

where $|\vec{n}|$ is fixed.
Let $Y = Y_{(n_1,n_2)}$ be the solution of RH problem (4.20)-(4.18), with $V(x) = \frac{1}{2}x^2$ and ω_1 and ω_2 given by (4.26).
Let also $P_{(n_1,n_2)}(x) = x^{|\vec{n}|} + ...$ be a monic polynomial of degree $|\vec{n}| = n_1 + n_2$ such that for $k = 1, 2$, one has

$$\int_{-\infty}^{\infty} P_{(n_1,n_2)}(x)x^j\omega_k(x)dx = 0, \quad j = 0, 1, ..., n_k - 1. \tag{4.27}$$

The polynomial $P_{(n_1,n_2)}(x)$ is unique and is called **multiple Hermite polynomial**

Denote that for $k = 1, 2$

$$h^{(k)}_{(n_1,n_2)} = \int_{-\infty}^{\infty} P_{(n_1,n_2)}(x) x^{n_k} \omega_k(x) dx \neq 0. \tag{4.28}$$

As in (4.19), we can rewrite the solution of the RH problem as

$$Y_{(n_1,n_2)} = \begin{pmatrix} P_{(n_1,n_2)} & C(P_{(n_1,n_2)}\omega_1) & C(P_{(n_1,n_2)}\omega_2) \\ c_1 P_{(n_1-1,n_2)} & c_1 C(P_{(n_1-1,n_2)}\omega_1) & c_1 C(P_{(n_1-1,n_2)}\omega_2) \\ c_2 P_{(n_1,n_2-1)} & c_2 C(P_{(n_1,n_2-1)}\omega_1) & c_2 C(P_{(n_1,n_2-1)}\omega_2) \end{pmatrix} \tag{4.29}$$

with constants

$$c_1 = -\frac{2\pi i}{h^{(1)}_{(n_1-1,n_2)}} \quad \text{and} \quad c_2 = -\frac{2\pi i}{h^{(1)}_{(n_1,n_2-1)}} \tag{4.30}$$

and where Cf denotes the Cauchy transform of f

$$Cf(z) = \frac{1}{2\pi i} \int \frac{f(s)}{s-z} ds. \tag{4.31}$$

The recurrence relations and differential equations are formulated in terms of function

$$\begin{aligned} \psi_{(n_1,n_2)} &= \begin{pmatrix} P_{(n_1,n_2)} e^{-\frac{1}{2}|\vec{n}|z^2} & C(P_{(n_1,n_2)}\omega_1) e^{-|\vec{n}|az} & C(P_{(n_1,n_2)}\omega_2) e^{|\vec{n}|az} \\ P_{(n_1-1,n_2)} e^{-\frac{1}{2}|\vec{n}|z^2} & C(P_{(n_1-1,n_2)}\omega_1) e^{-|\vec{n}|az} & C(P_{(n_1-1,n_2)}\omega_2) e^{|\vec{n}|az} \\ P_{(n_1,n_2-1)} e^{-\frac{1}{2}|\vec{n}|z^2} & C(P_{(n_1,n_2-1)}\omega_1) e^{-|\vec{n}|az} & C(P_{(n_1,n_2-1)}\omega_2) e^{|\vec{n}|az} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c_1} & 0 \\ 0 & 0 & \frac{1}{c_2} \end{pmatrix} Y_{(n_1,n_2)} \begin{pmatrix} e^{-\frac{1}{2}|\vec{n}|z^2} & 0 & 0 \\ 0 & e^{-|\vec{n}|az} & 0 \\ 0 & 0 & e^{|\vec{n}|az} \end{pmatrix}. \end{aligned} \tag{4.32}$$

The function $\psi = \psi_{(n_1,n_2)}$ solves the following RH problem:

- ψ is analytic on $\mathbb{C} \setminus \mathbb{R}$
- for $x \in \mathbb{R}$, we have

$$\psi_+(x) = \psi_-(x) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.33}$$

- as $z \to \infty$, we have the asymptotic behavior

$$\psi(z) = \left(I + O(\tfrac{1}{z})\right) \begin{pmatrix} z^{|\vec{n}|} e^{-\frac{1}{2}|\vec{n}|z^2} & 0 & 0 \\ 0 & \frac{1}{c_1} z^{-n_1} e^{-|\vec{n}|az} & 0 \\ 0 & 0 & \frac{1}{c_2} z^{-n_2} e^{|\vec{n}|az} \end{pmatrix}. \tag{4.34}$$

Property 4.3.2. We have the recurrence relations

$$\psi_{(n_1+1,n_2)}(z) = \begin{pmatrix} z-a & -\frac{n_1}{|\vec{n}|} & -\frac{n_2}{|\vec{n}|} \\ 1 & 0 & 0 \\ 1 & 0 & -2a \end{pmatrix} \psi_{n_1,n_2}(z) \tag{4.35}$$

$$\psi_{(n_1,n_2+1)}(z) = \begin{pmatrix} z+a & -\frac{n_1}{|\vec{n}|} & -\frac{n_2}{|\vec{n}|} \\ 1 & 2a & 0 \\ 1 & 0 & 0 \end{pmatrix} \psi_{n_1,n_2}(z) \tag{4.36}$$

and the differential equation

$$\psi'_{(n_1,n_2)}(z) = |\vec{n}| \begin{pmatrix} -z & \frac{n_1}{|\vec{n}|} & \frac{n_2}{|\vec{n}|} \\ -1 & -a & 0 \\ -1 & 0 & a \end{pmatrix} \psi_{n_1,n_2}(z). \tag{4.37}$$

The proof of this proposition is given after Section 4.3.3 because it needs some preparations.

Recurrence equations of multiple Hermite polynomials

Recall that $(k = 1, 2)$

- $$q^{(k)}_{(n_1,n_2)} = \int_{\infty}^{\infty} P_{(n_1,n_2)}(z) z^{n_k+1} \omega_k(z) dz$$

  then

  $$\int_{\infty}^{\infty} \frac{P_{(n_1,n_2)}(z)\omega_k(z)}{z} dz = \frac{q^{(k)}_{(n_1,n_2)}}{z^{n_k+2}}$$

- $$h^{(k)}_{(n_1,n_2)} = \int_{\infty}^{\infty} P_{(n_1,n_2)}(z) z^{n_k} \omega_k(z) dz$$

  then

  $$\int_{\infty}^{\infty} \frac{P_{(n_1,n_2)}(z)\omega_k(z)}{z} dz = \frac{h^{(k)}_{(n_1,n_2)}}{z^{n_k+1}}$$

- $$\frac{1}{u-z} = -\frac{1}{z}\sum_{k=0}^{\infty}(\frac{u}{z})^k = -(\frac{1}{z} + \frac{u}{z^2} + \frac{u^2}{z^3} + \ldots).$$

From the orthogonality equation (4.27) we obtain as $z \to \infty$

$$\begin{aligned} \frac{1}{2\pi i}\int_{\infty}^{\infty} \frac{P_{(n_1,n_2)}(u)\omega_k(u)}{u-z} du &= -\frac{1}{2\pi i}\int_{\infty}^{\infty} P_{(n_1,n_2)}(u)\omega_k(u)(\frac{1}{z} + \frac{u}{z^2} + \frac{u^2}{z^3} + \ldots) \\ &= -\frac{1}{2\pi i}(\frac{h^{(k)}_{(n_1,n_2)}}{z^{n_k+1}} + \frac{q^{(k)}_{(n_1,n_2)}}{z^{n_k+2}} + \ldots). \end{aligned} \tag{4.38}$$

This implies that

$$\Psi_{(n_1,n_2)}(z) = (I + \frac{\Psi^{(1)}_{(n_1,n_2)}}{z} + \ldots)\mathrm{diag}(z^{|\vec{n}|}e^{-\frac{1}{2}|\vec{n}|z^2}, \frac{e^{-|\vec{n}|az}}{c_1 z^{n_1}}, \frac{e^{|\vec{n}|az}}{c_2 z^{n_2}}) \tag{4.39}$$

with constants

$$c_1 = -\frac{2\pi i}{h^{(1)}_{(n_1-1,n_2)}} \quad \text{and} \quad c_2 = -\frac{2\pi i}{h^{(1)}_{(n_1,n_2-1)}} \tag{4.40}$$

and where

$$\Psi^{(1)}_{(n_1,n_2)} = \begin{pmatrix} p_{(n_1,n_2)} & \frac{h^{(1)}_{(n_1,n_2)}}{h^{(1)}_{(n_1-1,n_2)}} & \frac{h^{(2)}_{(n_1,n_2)}}{h^{(2)}_{(n_1,n_2-1)}} \\ 1 & \frac{q^{(1)}_{(n_1,n_2)}}{h^{(1)}_{(n_1-1,n_2)}} & \frac{h^{(2)}_{(n_1-1,n_2)}}{h^{(2)}_{(n_1,n_2-1)}} \\ 1 & \frac{h^{(1)}_{(n_1,n_2-1)}}{h^{(1)}_{(n_1-1,n_2)}} & \frac{q^{(2)}_{(n_1,n_2-1)}}{h^{(2)}_{(n_1,n_2-1)}} \end{pmatrix} \tag{4.41}$$

and

$$P_{(n_1,n_2)}(z) = z^{|\vec{n}|} + p_{(n_1,n_2)}(z)z^{|\vec{n}|-1} + \ldots.$$

Set

$$U_{(n_1,n_2)}(z) = \Psi_{(n_1+1,n_2)}(z)\Psi_{(n_1,n_2)}(z)^{-1}, \tag{4.42}$$

then by (4.34) there is no jump on the real line and we have

$$U_{(n_1,n_2)+}(x) = U_{(n_1,n_2)-}(x)$$

and as $z \to \infty$

$$\begin{aligned} U_{(n_1,n_2)}(z) &\cong \left(I + \tfrac{\Psi^{(1)}_{(n_1+1,n_2)}}{z} + \ldots\right) \begin{pmatrix} z & 0 & 0 \\ 0 & \frac{h^{(1)}_{(n_1,n_2)}}{h^{(1)}_{(n_1-1,n_2-1)}} & 0 \\ 0 & 0 & \frac{h^{(2)}_{(n_1+1,n_2-1)}}{h^{(2)}_{(n_1,n_2-1)}} \end{pmatrix} \left(I + \tfrac{\Psi^{(1)}_{(n_1,n_2)}}{z} + \ldots\right)^{-1} \\ &= zP_1 + \Psi^{(1)}_{(n_1+1,n_2)}P_1 - P_1\Psi^{(1)}_{(n_1,n_2)} + \frac{h^{(2)}_{(n_1+1,n_2-1)}}{h^{(2)}_{(n_1,n_2-1)}}P_3 + O(\tfrac{1}{z}) \end{aligned} \tag{4.43}$$

where

$$P_1 = \mathrm{diag}(1,0,0),\ P_2 = \mathrm{diag}(0,1,0),\ P_1 = \mathrm{diag}(0,0,1). \tag{4.44}$$

Since $U_{(n_1,n_2)}(z)$ is analytic in the complex plane, equation (4.43) implies by the Liouville theorem, that

$$\begin{aligned} U_{(n_1,n_2)}(z) &= zP_1 + \Psi^{(1)}_{(n_1+1,n_2)}P_1 - P_1\Psi^{(1)}_{(n_1,n_2)} + \frac{h^{(2)}_{(n_1+1,n_2-1)}}{h^{(2)}_{(n_1,n_2-1)}}P_3 \\ &= \begin{pmatrix} z - b_{(n_1,n_2)} & -c_{(n_1,n_2)} & -d_{(n_1,n_2)} \\ 1 & 0 & 0 \\ 1 & 0 & e_{(n_1,n_2)} \end{pmatrix}, \end{aligned} \tag{4.45}$$

where

$$c_{(n_1,n_2)} = \frac{h^{(1)}_{(n_1,n_2)}}{h^{(1)}_{(n_1-1,n_2)}} \neq 0,\ d_{(n_1,n_2)} = \frac{h^{(2)}_{(n_1,n_2)}}{h^{(2)}_{(n_1,n_2-1)}} \neq 0,\ e_{(n_1,n_2)} = \frac{h^{(2)}_{(n_1+1,n_2-1)}}{h^{(2)}_{(n_1,n_2-1)}} \neq 0. \tag{4.46}$$

Thus, we obtain the matrix recurrence equation

$$\Psi_{(n_1+1,n_2)}(z) = U_{(n_1,n_2)}(z)\Psi_{(n_1,n_2)}(z). \tag{4.47}$$

By restricting it to the $(1,1)$ entry we obtain that

$$P_{(n_1+1,n_2)}(z) = (z - b_{(n_1,n_2)}P_{(n_1,n_2)}(z) - c_{(n_1,n_2)}P_{(n_1-1,n_2)}(z) - d_{(n_1,n_2)}P_{(n_1,n_2-1)}(z) \tag{4.48}$$

and by restricting it to $(3,1)$ entry we obtain that

$$P_{(n_1+1,n_2-1)}(z) = P_{(n_1,n_2)}(z) - e_{(n_1,n_2)}P_{(n_1,n_2)}(z) \tag{4.49}$$

We have another recurrence equation similar to (4.47)

$$\Psi_{(n_1,n_2+1)}(z) = \tilde{U}_{(n_1,n_2)}(z)\Psi_{(n_1,n_2)}(z) \tag{4.50}$$

where

$$\tilde{U}_{(n_1,n_2)}(z) = \begin{pmatrix} z - \tilde{b}_{(n_1,n_2)} & -c_{(n_1,n_2)} & -d_{(n_1,n_2)} \\ 1 & \tilde{e}_{(n_1,n_2)} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.51}$$

and

$$\tilde{e}_{(n_1,n_2)} = \frac{h^{(1)}_{(n_1-1,n_2+1)}}{h^{(1)}_{(n_1-1,n_2)}} \neq 0 \tag{4.52}$$

By restricting (4.50) to $(1,1)$ and $(2,1)$ entry we obtain respectively the equations

$$P_{(n_1,n_2+1)}(z) = (z - \tilde{b}_{(n_1,n_2)}P_{(n_1,n_2)}(z) - c_{(n_1,n_2)}P_{(n_1-1,n_2)}(z) - d_{(n_1,n_2)}P_{(n_1,n_2-1)}(z) \tag{4.53}$$

and

$$P_{(n_1-1,n_2+1)}(z) = P_{(n_1,n_2)}(z) - \tilde{e}_{(n_1,n_2)}P_{(n_1-1,n_2)}(z) \tag{4.54}$$

Differential equation for multiple Hermite polynomials

From (4.37) we have

$$A_{(n_1,n_2)}(z) = \frac{1}{|\vec{n}|}\Psi'_{(n_1,n_2)}(z)\Psi_{(n_1,n_2)}(z)^{-1} \tag{4.55}$$

It follows from (4.34) that $A_{(n_1,n_2)}(z)$ has no jump on the real axis, so that it is analytic on the complex plane.
By differentiating (4.39) we obtain as $z \to \infty$,

$$A_{(n_1,n_2)}(z) = \left(I + \frac{\Psi^{(1)}_{(n_1+1,n_2)}}{z} + \dots\right) \begin{pmatrix} -z & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \left(I + \frac{\Psi^{(1)}_{(n_1,n_2)}}{z} + \dots\right)^{-1} + O(\frac{1}{z}) \tag{4.56}$$

Since $A_{(n_1,n_2)}(z)$ is analytic, we obtain that

$$A_{(n_1,n_2)}(z) = -\Big[\Big(I + \tfrac{\Psi^{(1)}_{(n_1+1,n_2)}}{z} + \dots\Big)\begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Big(I + \tfrac{\Psi^{(1)}_{(n_1,n_2)}}{z} + \dots\Big)^{-1}\Big]_{pol} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \tag{4.57}$$

where $[f(z)]_{pol}$ means the polynomial part of $f(z)$ at infinity.
From (4.55) we get the differential equation

$$\Psi'_{(n_1,n_2)}(z) = |\vec{n}| A_{(n_1,n_2)}(z)\Psi_{(n_1,n_2)}(z) \tag{4.58}$$

and (4.57) reduces to

$$A_{(n_1,n_2)}(z) = \begin{pmatrix} -z & c_{(n_1,n_2)} & d_{(n_1,n_2)} \\ -1 & -a & 0 \\ -1 & 0 & a \end{pmatrix}. \tag{4.59}$$

We can then prove Propostion 4.3.2

Proof of proposition 4.3.2

Proof. From equation (4.47),we have by appling the derivative of a product

$$U'_{(n_1,n_2)}(z) = \Psi'_{(n_1+1,n_2)}(z)\Psi^{-1}_{(n_1,n_2)}(z) + \Psi_{(n_1+1,n_2)}(z)\Psi'^{-1}_{(n_1,n_2)}(z). \tag{4.60}$$

Using (4.58) in (4.60) and applying the derivative of a ratio we have

$$U'_{(n_1,n_2)}(z) = |\vec{n}| A_{(n_1+1,n_2)}(z)\Psi_{(n_1+1,n_2)}(z)\Psi^{-1}_{(n_1,n_2)}(z) - |\vec{n}|\Psi_{(n_1+1,n_2)}(z)\Psi^{-1}_{(n_1,n_2)}(z)A_{(n_1,n_2)}(z)$$

which by (4.47) gives

$$\frac{1}{|\vec{n}|}U'_{(n_1,n_2)}(z) = A_{(n_1+1,n_2)}(z)U_{(n_1,n_2)}(z) - U_{(n_1,n_2)}(z)A_{(n_1,n_2)}(z).$$

In the same way, (4.50) and (4.58) give

$$\frac{1}{|\vec{n}|}\tilde{U}'_{(n_1,n_2)}(z) = A_{(n_1,n_2+1)}(z)\tilde{U}_{(n_1,n_2)}(z) - \tilde{U}_{(n_1,n_2)}(z)A_{(n_1,n_2)}(z). \tag{4.61}$$

Both equations give

$$
\begin{aligned}
&b_{(n_1,n_2)} = a, \\
&\tilde{b}_{(n_1,n_2)} = -a, \\
&c_{(n_1+1,n_2)} = c_{(n_1,n_2)} + \frac{1}{|\vec{n}|}, \\
&c_{(n_1,n_2+1)} = c_{(n_1,n_2)}, \\
&d_{(n_1+1,n_2)} = d_{(n_1,n_2)}, \\
&d_{(n_1,n_2+1)} = d_{(n_1,n_2)} + \frac{1}{|\vec{n}|}, \\
&e_{(n_1,n_2)} = -2a, \\
&\tilde{e}_{(n_1,n_2)} = 2a.
\end{aligned}
\tag{4.62}
$$

Since $c_{(0,n_2)} = d_{(n_1,0)} = 0$, we obtain that

$$c_{(n_1,n_2)} = \frac{n_1}{|\vec{n}|} \tag{4.63}$$

and

$$e_{(n_1,n_2)} = \frac{n_2}{|\vec{n}|}. \tag{4.64}$$

Equations (4.63) and (4.64) prove the equation (4.35) and (4.37).
Similarly we obtain that

$$\tilde{e}_{(n_1,n_2)} = 2a \tag{4.65}$$

and this proves the equation (4.36). □

Chapter 5

Steepest Descent method of Deift-Zhou

The main references used in this chapter are [14] and [10].

The aim of this chapter is to analyze the RH problem as $z \to \infty$ by using the steepest descent method of Deift and Zhou. This method consists of transforming the initial RH problem Y in a few steps to another equivalent RH problem for a matrix valued function R.This matrix R is analytic in $\mathbb{C} \setminus \mathbb{R}$ and normalized at infinity, the jumps on each the contours Γ in $\mathbb{R}$ are uniformly close to the identity matrix.
One can then conclude that the solution R of this model RH problem will be close to the identity matrix. By reversing the steps we can then go back from R to the original matrix Y.

5.1 Riemann surface

We look for a solution of the differential equation (4.37) of the form

$$\psi_{(n_1,n_2)}(z) = W(z)e^{-|\vec{n}|\Lambda(z)} \tag{5.1}$$

where Λ is a diagonal matrix.
Then

$$\psi'_{(n_1,n_2)}(z) = W'e^{-|\vec{n}|\Lambda(z)} - |\vec{n}|W\Lambda'e^{-|\vec{n}|\Lambda(z)}.$$

Then equation (4.37) becomes successively

$$\begin{aligned}
W'e^{-|\vec{n}|\Lambda(z)} - |\vec{n}|W\Lambda' e^{-|\vec{n}|\Lambda(z)} &= |\vec{n}|AW(z)e^{-|\vec{n}|\Lambda(z)} \\
W' - |\vec{n}|W\Lambda' &= |\vec{n}|AW \\
\frac{W'}{|\vec{n}|} - W\Lambda' &= AW \\
\frac{W'}{|\vec{n}|}W^{-1} - W\Lambda' W^{-1} &= A \\
-W\Lambda' W^{-1} &= A - \frac{1}{|\vec{n}|}W'W^{-1} \qquad (5.2)
\end{aligned}$$

where A is the matrix of coefficients in (4.37).
By dropping the last term in (5.2), we reduce it to the eigenvalue problem

$$W\Lambda' W^{-1} = -A. \qquad (5.3)$$

The characteric polynomial is

$$\det(\xi I + A) = \begin{vmatrix} \xi - z & t_1 & t_2 \\ -1 & \xi - a & 0 \\ -1 & 0 & \xi + a \end{vmatrix} = \xi^3 - z\xi^2 + (t_1 + t_2 - a^2)\xi + (t_1 - t_2 + za)a \qquad (5.4)$$

where $t_1 = \frac{n_1}{|\vec{n}|}$ and $t_2 = \frac{n_2}{|\vec{n}|}$.
The spectral cubic equation

$$\xi^3 - z\xi^2 + (t_1 + t_2 - a^2)\xi + (t_1 - t_2 + za)a = 0$$

defines a Riemann surface, which in the case of $n_1 = n_2 = \frac{|\vec{n}|}{2}$ reduces to

$$\xi^3 - z\xi^2 - (a^2 - 1)\xi + za^2 = 0. \qquad (5.5)$$

This define the Riemann surface which is given by

$$\xi^3 - z\xi^2 - (a^2 - 1)\xi + za^2 = 0. \qquad (5.6)$$

If we solve for z, one has

$$z = \frac{\xi^3 - (a^2 - 1)\xi}{\xi^2 - a^2}. \qquad (5.7)$$

One needs the three solutions $\xi_1(z), \xi_2(z), \xi_3(z)$ of the cubic equation(5.6) numbered according to their asymptotic behavior as $z \to \infty$

$$\begin{aligned}
\xi_1(z) =& z - \frac{1}{z} + O(\frac{1}{z^3}) \\
\xi_2(z) =& a + \frac{1}{2z} + O(\frac{1}{z^2}) \qquad (5.8) \\
\xi_3(z) =& -a + \frac{1}{2z} + O(\frac{1}{z^2}).
\end{aligned}$$

The critical points of $\xi(z)$ satisfy the equation

$$\xi^4 - (1+2a)\xi^2 + (a^2-1)a^2 = 0 \tag{5.9}$$

which is biquadratic.
Its roots are

$$\xi_{1,2}^2 = \frac{1}{2} + a^2 \pm \frac{1}{2}\sqrt{1+8a^2}. \tag{5.10}$$

Consider the case $a > 1$,all roots are real.

$$p, q = \sqrt{\frac{1}{2} + a^2 \mp \frac{1}{2}\sqrt{1+8a^2}} \qquad 0 < p < q. \tag{5.11}$$

The critical points are then $\xi = \pm p, \pm q$. The branch points on the z-plane are $\pm z_1$ and $\pm z_2$, where substiting (5.11) in (5.7)

$$z_1 = q\frac{\sqrt{1+8a^2}+3}{\sqrt{1+8a^2}+1} \tag{5.12}$$

$$z_2 = q\frac{\sqrt{1+8a^2}-3}{\sqrt{1+8a^2}-1}, \qquad 0 < z_2 < z_1 \tag{5.13}$$

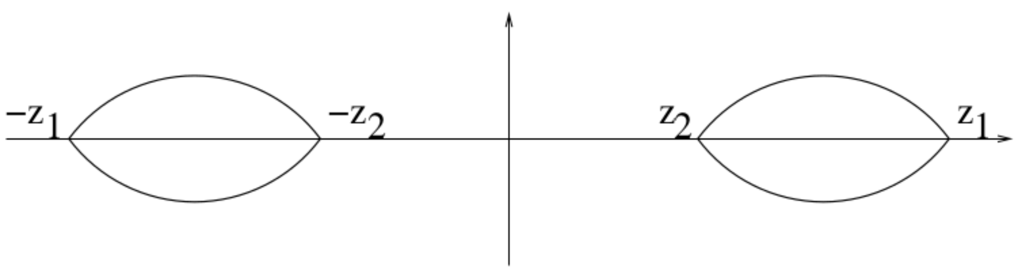

Figure 5.1: The lenses with vertices $-z_1, -z_2$ and z_2, z_1

We have a proposition proved in [14].

Property 5.1.1. We have $\rho(x) > 0$ for $x \in (-z_1, -z_2) \cup (z_1, z_2)$ and

$$\int_{-z_1}^{-z_2} \rho(x)dx = \int_{z_1}^{z_2} \rho(x)dx = \frac{1}{2}. \tag{5.14}$$

Moreover, the are $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned} \rho(x) &= \frac{\rho_j}{\pi}|x-z_j|^{\frac{1}{2}}(1+O(x-z_j)) \quad \text{as } x \to z_j,\ x \in (z_2, z_1) \\ \rho(x) &= \frac{\rho_j}{\pi}|x+z_j|^{\frac{1}{2}}(1+O(x+z_j)) \quad \text{as } x \to -z_j,\ x \in (-z_1, -z_2). \end{aligned} \tag{5.15}$$

We need the integral of the $\xi-$functions

$$\lambda_k(z) = \int^z \xi_k(s)ds = \int_{z_1}^{z_2} \xi_k(s)ds, \quad k = 1,2,3 \tag{5.16}$$

Using (5.8) in (5.16), it follows respectively that as $z \to \infty$

$$\begin{aligned} \lambda_1(z) &= \int^z \xi_1(s)ds \\ &= \int^z \Big[z - \frac{1}{z} + O(\frac{1}{z^3})\Big]ds \\ &= \frac{z^2}{2} - lnz + l_1 + O(\frac{1}{z^2}) \end{aligned} \tag{5.17}$$

Similarly

$$\lambda_2(z) = az + \frac{1}{2}lnz + l_2 + O(\frac{1}{z}) \tag{5.18}$$

and

$$\lambda_3(z) = -az + \frac{1}{2}lnz + l_3 + O(\frac{1}{z}) \tag{5.19}$$

where l_1, l_2 and l_3 are some constants. From the Riemann surface we obtain the functions ξ_j and λ_j, $j = 1,2,,3$ that are necessary for the transformations of the RH problem.

5.2 Tranformations of the RH problem

One of the main advantages of the RH approach for multiple orthogonal polynomials is that this is a very useful setting to obtain uniform asymptotics valid in the whole complex plane. The idea is to transform the initial RH problem Y in a few steps to another equivalent RH problem for a matrix valued function R which is analytic in $\mathbb{C} \setminus \mathbb{R}$. This new RH is normalized at infinity, so that $R(z) = I + O(1/z)$ as $z \to \infty$, and the jumps on each the contours Γ in $\mathbb{R}$ are uniformly close to the identity matrix.
One can then conclude that the solution R of this model RH problem will be close to the identity matrix

$$R(z) = I + O(\frac{1}{|\vec{n}|}), \ |\vec{n}| \to \infty$$

uniformly for $z \in \mathbb{C}$. By reversing the steps we can then go back from R to the original matrix Y and read of the required asymptotic behavior as $|\vec{n}| \to \infty$. The transformation from Y to R goes as follows:

1. Transform Y to T such that T satisfies a RH problem with a simple jump on $\mathbb{R}$ and such that T is normalized at infinity:

$$T(z) = I + O(\frac{1}{z}), \text{ as } z \to \infty.$$

This step requires detailed knowledge of the asymptotic zero distribution of the orthogonal polynomials and uses relevant properties of the logarithmic potential of this zero distribution. The jump matrix on $\mathbb{R}$ will contain oscillatory terms on the interval where the zeros are dense,

2. Transform T to S such that S is still normalized at infinity but we deform the contour R to a collection of contours such that the jumps of S on each contour in those contours are no longer oscillatory. This deformation is similar to a contour deformation in the steepest descent method for obtaining asymptotics of an oscillatory integral and hence this is known as a steepest descent method for RH problems. It was first developed by Deift and Zhou in 1993,

3. Some of the jumps for S are close to the identity matrix. Ignoring these jumps, one arrives at a normalized RH problem for P. This P is expected to be close to S as $|\vec{n}| \to \infty$ and it will be called the parametrix for the outer region,

4. At the endpoints and at the intersection points of the contours the jumps for S will usually not be close to the identity matrix. Around these points z_k we need to make a local analysis of the RH. Around each endpoint or intersection point z_k we need to construct a local parametrix P_k , which is the solution of a RH problem with jumps on the contours in the neighborhood of the point z_k under investigation and such that this P_k matches the parametrix P on a contour Γ_k encircling z_k up to terms of order $O(\frac{1}{z})$,

5. Transform S to R by setting $R = SP^{-1}$ away from the points z_k , and $R = SP_k^{-1}$ in the neighborhood of z_k . This R will then be normalized at ∞ and it will have jumps on a collection of contours which contains parts of the contours and the contours Γ_k encircling the endpoints/intersection points z_k . All these jumps are uniformly close to the identity matrix.

5.2.1 First transformation of the RH problem: $Y \to T$

Using the functions λ_j and the constants $c_j,\ \ j = 1, 2, 3$ defined in the previous section, we define

$$T(z) = diag(e^{-|\vec{n}|l_1}, e^{-|\vec{n}|l_2}, e^{-|\vec{n}|l_3})Y(z)diag(e^{|\vec{n}|(\lambda_1(z)-\frac{1}{2}z^2)}, e^{|\vec{n}|(\lambda_2(z)-az)}, e^{|\vec{n}|(\lambda_3(z)+az)}). \quad (5.20)$$

Then by (4.16) and (5.20) we have

$$T_+(x) = T_-(x)jT(x) \quad (5.21)$$

where

$$jT(x) = \begin{pmatrix} e^{|\vec{n}|(\lambda_{1+}(x)-\lambda_{1-}(x))} & e^{|\vec{n}|(\lambda_{2+}(x)-\lambda_{1-}(x))} & e^{|\vec{n}|(\lambda_{3+}(x)-\lambda_{1-}(x))} \\ 0 & e^{|\vec{n}|(\lambda_{2+}(x)-\lambda_{2-}(x))} & 0 \\ 0 & 0 & e^{|\vec{n}|(\lambda_{3+}(x)-\lambda_{3-}(x))} \end{pmatrix}. \quad (5.22)$$

We simplify the jump matrix jT on the different parts on the real axis:

1. on $[z_2, z_1]$, (5.22) reduces to

$$jT(x) = \begin{pmatrix} e^{|\vec{n}|(\lambda_1-\lambda_2)_+} & 1 & e^{|\vec{n}|(\lambda_{3+}-\lambda_{1-})} \\ 0 & e^{|\vec{n}|(\lambda_1-\lambda_2)_-} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.23}$$

2. on $[-z_1, -z_2]$, (5.22) reduces to

$$jT(x) = \begin{pmatrix} e^{|\vec{n}|(\lambda_1-\lambda_3)_+} & e^{|\vec{n}|(\lambda_{2+}-\lambda_{1-})} & 1 \\ 0 & 1 & e^{|\vec{n}|(\lambda_1-\lambda_3)_-} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.24}$$

3. on $(-\infty, -z_1] \cup [-z_2, z_2] \cup [z_1, \infty]$, (5.22) reduces to

$$jT(x) = \begin{pmatrix} 1 & e^{|\vec{n}|(\lambda_{2+}-\lambda_{1-})} & e^{|\vec{n}|(\lambda_{3+}-\lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.25}$$

The asymptotic behaviors of T are:

$$T(z) = I + O(\frac{1}{z}), \text{ as } z \to \infty. \tag{5.26}$$

Thus T solves the following RH problem:

- T is analytic on $\mathbb{C} \setminus \mathbb{R}$
- $$T_+(x) = T_-(x) jT(x), \quad x \in \mathbb{R} \tag{5.27}$$
- as $z \to \infty$
$$T(z) = I + O(\frac{1}{z}). \tag{5.28}$$

Loading (5.20) in (4.24), the kernel is expressed in terms of T as follows

$$K_{\vec{n}}(x, y) = \frac{e^{-\frac{1}{4}|\vec{n}|(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{|\vec{n}|\lambda_{2+}(y)} & e^{|\vec{n}|\lambda_{3+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-|\vec{n}|\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix} \tag{5.29}$$

5.2.2 Second transformation of the RH problem: $T \to S$

Consider a lens with vertices z_2, z_1 (see fig 5.1).
The lens is contained in the neighbordhood of (z_2, z_1).

We have the factorisation

$$\begin{pmatrix} e^{|\vec{n}|(\lambda_1-\lambda_2)_+} & 1 & e^{|\vec{n}|(\lambda_{3+}-\lambda_{1-})} \\ 0 & e^{|\vec{n}|(\lambda_1-\lambda_2)_-} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)_-} & 1 & -e^{-|\vec{n}|(\lambda_3-\lambda_2)_-} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)_+} & 1 & -e^{-|\vec{n}|(\lambda_3-\lambda_2)_+} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.30}$$

Set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ -e^{|\vec{n}|(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{|\vec{n}|(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ in the upper lens region} \\ \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{|\vec{n}|(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ in the lower lens region} \end{cases} \tag{5.31}$$

Then (5.27) and (5.31) imply that

$$S_+(x) = S_-(x) jS(x);\ \ jS(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \ x \in [z_2, z_1] \tag{5.32}$$

Similarly, consider a lens with vertices $-z_1, -z_2$ and set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{|\vec{n}|(\lambda_1(z)-\lambda_3(z))} & -e^{|\vec{n}|(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ in the upper lens region} \\ \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_3(z))} & -e^{|\vec{n}|(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ in the lower lens region.} \end{cases} \tag{5.33}$$

Then (5.27) and (5.33) imply that

$$S_+(x) = S_-(x) jS(x);\ \ jS(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ \ x \in [-z_1, -z_2]. \tag{5.34}$$

Set

$$S(z) = T(z) \text{ outside the lens regions.} \tag{5.35}$$

Then we have jumps on the boundary of lenses

$$S_+(x) = S_-(x) jS(x) \tag{5.36}$$

where the contours are oriented from left to right(that is from $-z_1$ to $-z_2$ or from z_2 to z_1) and where $S_+(S_-)$ denote the limiting value of S from left(right) if we traverse the contour according to its orientation.

The jump matrix jS in (5.36) has the form

$$\begin{aligned}
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_2(z))} & 1 & e^{|\vec{n}|(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the upper boundary of the } [z_2; z_1]-\text{lens} \\
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{|\vec{n}|(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the lower boundary of the } [z_2; z_1]-\text{lens} \\
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_3(z))} & e^{|\vec{n}|(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ on the upper boundary of the } [-z_1; -z_2]-\text{lens} \\
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{|\vec{n}|(\lambda_1(z)-\lambda_3(z))} & -e^{|\vec{n}|(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ on the lower boundary of the } [-z_1; -z_2]-\text{lens}
\end{aligned} \tag{5.37}$$

On $(-\infty, -z_1] \cup [-z_2, z_2] \cup [z_1, \infty)$, S has the same jump as T, so that

$$S_+(x) = S_-(x) jS(x), \;\; jS(x) = jT(x), \;\; x \in (-\infty, -z_1] \cup [-z_2, z_2] \cup [z_1, \infty). \tag{5.38}$$

Thus S solves the following RH problem

- S is analytic on $\mathbb{C} \setminus (\mathbb{R} \cup \Gamma)$, where Γ is the boundary of lenses,

$$S_+(z) = S_-(z) jS(z), \;\; z \in \mathbb{R} \cup \Gamma \tag{5.39}$$

- as $z \to \infty$,

$$S(z) = I + O(\frac{1}{z}). \tag{5.40}$$

Consider the function h defined on $(-z_1, -z_2) \cup (z_2, z_1)$:

$$h(x) = -\frac{1}{2}x^2 + Re\lambda_{1+}(x), \;\; x \in (-z_1, -z_2) \cup (z_2, z_1) \tag{5.41}$$

with $\lambda_{1+}(x) = \int_{z_1}^{x} \xi_{1+}(s)ds$, where ξ_{1+} is defined in (5.8).

The kerneL $K_{\vec{n}}$ is then expressed in terms of S as follows, see (5.29), (5.31) and (5.33). For x and y in (z_2, z_1) we have

$$K_{\vec{n}}(x,y) = \frac{e^{-\frac{1}{4}|\vec{n}|(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -e^{|\vec{n}|\lambda_{1+}(y)} & e^{|\vec{n}|\lambda_{2+}(y)} & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-|\vec{n}|\lambda_{1+}(x)} \\ e^{-|\vec{n}|\lambda_{2+}(x)} \\ 0 \end{pmatrix} \tag{5.42}$$

while for x and y in $(-z_1, -z_2)$ we have

$$K_{\vec{n}}(x,y) = \frac{e^{-\frac{1}{4}|\vec{n}|(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -e^{|\vec{n}|\lambda_{1+}(y)} & 0 & e^{|\vec{n}|\lambda_{3+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-|\vec{n}|\lambda_{1+}(x)} \\ 0 \\ e^{-|\vec{n}|\lambda_{3+}(x)} \end{pmatrix} \tag{5.43}$$

since λ_1 and λ_2 are conjugates on (z_2, z_1).
We can rewrite (5.42) for $x, y \in (z_2, z_1)$ as

$$K_{\vec{n}}(x,y) = \frac{e^{|\vec{n}|(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{|\vec{n}|iIm\lambda_{1+}(y)} & 0 & e^{-|\vec{n}|iIm\lambda_{1+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-|\vec{n}|iIm\lambda_{1+}(y)} \\ e^{|\vec{n}|iIm\lambda_{1+}(y)} \\ 0 \end{pmatrix} \tag{5.44}$$

where h is defined by (5.41).
Similarly we have for $x, y \in (-z_1, -z_2)$

$$K_{\vec{n}}(x,y) = \frac{e^{|\vec{n}|(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{|\vec{n}|iIm\lambda_{1+}(y)} & e^{-|\vec{n}|iIm\lambda_{1+}(y)} & 0 \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-|\vec{n}|iIm\lambda_{1+}(y)} \\ 0 \\ e^{|\vec{n}|iIm\lambda_{1+}(y)} \end{pmatrix}. \tag{5.45}$$

5.2.3 Model RH problem

From (5.37) for z on the boundary of the lenses, and from (5.38) and (5.23) for z on the real intervals $(-\infty, -z_1), (-z_2, z_2)$ and (z_1, ∞), it follows that:
as $|\vec{n}| \to \infty$, the jump matrix $jS(z)$ is exponentially close to identity matrix at every z outside of $[-z_1, -z_2] \cup [z_2, z_1]$.
In this sub-section we solve the following model RH problem,where we ignore the exponentially jumps:
find $M : \mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1]) \to \mathbb{C}^{3\times 3}$ such that

- M is analytic on $\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])$
- for $x \in \mathbb{C} \setminus (-z_1, -z_2) \cup (z_2, z_1)$,

$$M_+(x) = M_-(x) jS(x) \tag{5.46}$$

- as $z \to \infty$,

$$M(z) = I + O(\frac{1}{z}). \tag{5.47}$$

We rift the model RH problem to the Riemann surface of (5.6). We consider the range of the functions ξ_k on the complex plane

$$\begin{aligned} \Omega_1 &= \xi_1(\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])), \\ \Omega_2 &= \xi_2(\mathbb{C} \setminus [z_2, z_1]), \\ \Omega_3 &= \xi_3(\mathbb{C} \setminus [-z_1, -z_2]). \end{aligned} \tag{5.48}$$

such that $a \in \Omega_2$ and $-a \in \Omega_3$.

We denote by Γ_k the boundary of $\Omega, k = 2, 3$. Then Ω_1, Ω_2 and Ω_3 give a partition of the

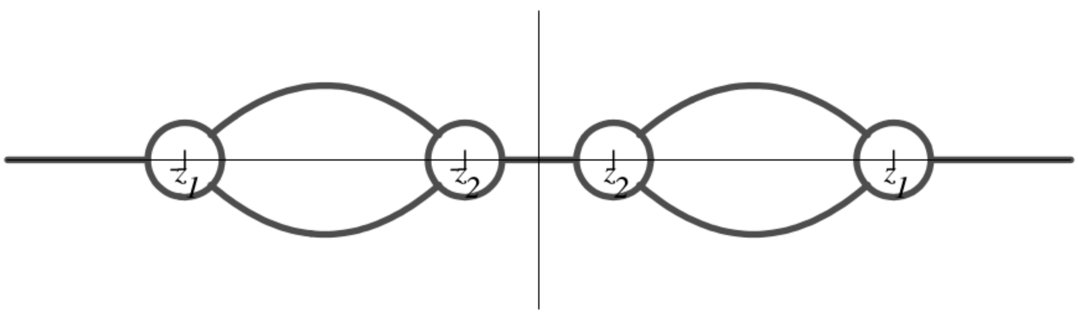

Figure 5.2: Contours Γ_R for R

complex plane into three regions(see [14]).

We look for a solution M in the following form:

$$M(z) = \begin{pmatrix} M_1(\xi_1(z)) & M_1(\xi_2(z)) & M_1(\xi_3(z)) \\ M_2(\xi_1(z)) & M_2(\xi_2(z)) & M_2(\xi_3(z)) \\ M_3(\xi_1(z)) & M_3(\xi_2(z)) & M_3(\xi_3(z)) \end{pmatrix} \tag{5.49}$$

Since $\xi_1(\infty) = \infty, \xi_2(\infty) = a, \xi_3(\infty) = -a$, then to satisfy(5.47) we demand [14]

$$\begin{aligned} &M_1(\infty) = 1,\ M_2(a) = 0,\ M_2(-a) = 0 \\ &M_2(\infty) = 0,\ M_2(a) = 1,\ M_3(-a) = 0 \\ &M_3(\infty) = 0,\ M_2(a) = 0,\ M_3(-a) = 1 \end{aligned} \tag{5.50}$$

Then we have the following solution:

$$M_1(\xi) = \frac{\xi^2 - a^2}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)}},\ M_{2,3}(\xi) = c_{2,3} \frac{\xi \pm a}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)}} \tag{5.51}$$

We have

$$(\xi^2 - p^2)(\xi^2 - q^2) = \xi^4 - (p^2 + q^2)\xi^2 + p^2q^2$$

where

$$p^2 = \frac{1}{2} + a^2 - \frac{1}{2}\sqrt{1 + 8a^2} \tag{5.52}$$

and

$$q^2 = \frac{1}{2} + a^2 + \frac{1}{2}\sqrt{1 + 8a^2} \tag{5.53}$$

$0 < p < q$.
It is easy to see that $p^2 + q^2 = 1 + 2a^2$ and $p^2q^2 = (a^2 - 1)a^2$.
Then by (5.9) we have

$$(\xi^2 - p^2)(\xi^2 - q^2) = \xi^4 - (1 + 2a^2)\xi^2 + (a^2 - 1)a^2 \tag{5.54}$$

Hence

$$M_2(a) = c_2 \frac{2a}{\sqrt{-2a^2}}. \tag{5.55}$$

By taking into account the cuts of $M_2(\xi)$ we obtain

$$M_2(-a) = c_2 i\sqrt{2} \tag{5.56}$$

hence

$$c_2 = -\frac{i}{\sqrt{2}} \tag{5.57}$$

Similary,

$$M_3(a) = c_3 \frac{2a}{\sqrt{-2a^2}} = c_3 i\sqrt{2} \tag{5.58}$$

hence

$$c_3 = c_2 = -\frac{i}{\sqrt{2}}. \tag{5.59}$$

Thus, the solution to the model RH problem is given by

$$M(z) = \begin{pmatrix} \frac{\xi_1^2(z)-a^2}{\sqrt{(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & \frac{\xi_2^2(z)-a^2}{\sqrt{(\xi_2^2(z)-p^2)(\xi_1^2(z)-q^2)}} & \frac{\xi_3^2(z)-a^2}{\sqrt{(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \\ -i\frac{\xi_1(z)+a}{\sqrt{2(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & -i\frac{\xi_2(z)+a}{\sqrt{2(\xi_2^2(z)-p^2)(\xi_2^2(z)-q^2)}} & -i\frac{\xi_3(z)+a}{\sqrt{2(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \\ -i\frac{\xi_1(z)-a}{\sqrt{2(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & -i\frac{\xi_2(z)-a}{\sqrt{2(\xi_2^2(z)-p^2)(\xi_2^2(z)-q^2)}} & -i\frac{\xi_3(z)-a}{\sqrt{2(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \end{pmatrix} \tag{5.60}$$

with cuts on $[z_2, z_1]$ and $[-z_1, -z_2]$.
The model solution $M(z)$ will be used to construct a *parametrix* for the RH problem for S outside of a neighborhood of the edge points.

5.2.4 Parametrix at edge points

We consider small disks $D(\pm z_j, r)$ with radius $r > 0$ and centered at the edge points, and look for a local parametrix P defined on the union of the four disks such that

- P is analytic on $D(\pm z_j, r) \setminus (\mathbb{R} \cup \Gamma)$,

$$P_+(z) = P_-(z) jS(z), \;\; z \in (\mathbb{R} \cup \Gamma) \cap D(\pm z_j, r), \tag{5.61}$$

- as $|\vec{n}| \to \infty$,

$$P(z) = \Big(I + O(\frac{1}{|\vec{n}|})\Big) M(z) \text{ uniformly for } z \in \partial D(\pm z_j, r). \tag{5.62}$$

We consider here the edge point z_1 in detail.
We note by (5.15) and (5.16) we have as $z \to \infty$,

$$\begin{aligned}
\lambda_1(z) &= q(z - z_1) + \frac{2\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^2, \\
\lambda_2(z) &= q(z - z_1) - \frac{2\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^2
\end{aligned} \tag{5.63}$$

so that

$$\lambda_1(z) - \lambda_2(z) = \frac{4\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^{5/2} \tag{5.64}$$

as $z \to z_1$.
The it follows that

$$\beta(z) = [\frac{3}{4}(\lambda_1(z) - \lambda_2(z))]^{2/3} \tag{5.65}$$

is analytic at z_1.
We take Γ near z_1 such that

$$\beta(\Gamma \cap D(z_1, r)) \subset \Big\{ z | arg(z) = \pm\frac{2\pi}{3} \Big\}.$$

Then Γ and $\mathbb{R}$ divide the disk $D(z_1, r)$ into four regions numbered I,II,III and IV such that

1. $0 < arg\beta(z) < 2\pi/3$, for z in region I
2. $2\pi/3 < arg\beta(z) < \pi$, for z in region II
3. $-\pi < arg\beta(z) < -2\pi/3$, for z in region III
4. $-2\pi/3 < arg\beta(z) < 0$, for z in region IV.

Recall that the jumps jS near z_1 are given by (5.32), (5.40) and (5.24):

$$\begin{aligned}
jS(z) &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } [z_1 - r, z_1], \\
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)} & 1 & e^{|\vec{n}|(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the upper boundary of the lens in } D(z_1, r), \\
jS(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)} & 1 & -e^{|\vec{n}|(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the lower boundary of the lens in } D(z_1, r), \\
jS(z) &= \begin{pmatrix} 1 & e^{|\vec{n}|(\lambda_2-\lambda_1)} & e^{|\vec{n}|(\lambda_3-\lambda_1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } [z_1, z_1 + r]. \qquad (5.66)
\end{aligned}$$

We write

$$\tilde{P} = \begin{cases} P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{|\vec{n}|(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} & \text{in regions I and IV} \\ P & \text{in regions II and III.} \end{cases} \qquad (5.67)$$

Then the jumps for $\tilde{P}$ are $\tilde{P}_+ = \tilde{P}_- j_{\tilde{P}}$, where

$$\begin{aligned}
j_{\tilde{P}} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } [z_1 - r, z_1], \\
j_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on the upper side of the lens in } D(z_1, r), \\
j_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 \\ e^{|\vec{n}|(\lambda_1-\lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on the lower side of the lens in } D(z_1, r), \\
j_{\tilde{P}} &= \begin{pmatrix} 1 & e^{|\vec{n}|(\lambda_2-\lambda_1)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ on } [z_1, z_1 + r]. \qquad (5.68)
\end{aligned}$$

We still have the matching condition

$$\tilde{P}(z) = \Big(I + O(\frac{1}{|\vec{n}|})\Big) M(z) \text{ uniformly for } z \in \partial D(z_1, r). \qquad (5.69)$$

The RH problem for $\tilde{P}$ is essentally a 2×2 problem, since the jumps (5.68) are non-trivial only in the upper 2×2 block.

Its solution can be constructed in a standard way out of Airy functions.
The Airy function $Ai(z)$ solves the equation

$$y^{''} = zy$$

and for any $\varepsilon > 0$, in the sector $\pi + \varepsilon \leq argz \leq \pi - \varepsilon$, it has the asymptotics as $z \to \infty$, see [14]

$$Ai(z) = \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{3}{2} z^{3/2}} \Big(1 + O(z^{-3/2})\Big). \tag{5.70}$$

The functions $Ai(\omega z), Ai(\omega^2 z)$, where $\omega = e^{\frac{2\pi}{3}}$, also solve the equation $y^{''} = zy$ and we the linear relation

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0. \tag{5.71}$$

Write

$$y_0(z) = Ai(z), y_1(z) = \omega Ai(\omega z), y_2(z) = \omega^2 Ai(\omega^2 z) = 0 \tag{5.72}$$

and we use these functions to define

$$\Phi(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) & 0 \\ y_0^{'}(z) & -y_2^{'}(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } 0 < argz < 2\pi/3 \\ \\ \begin{pmatrix} -y_1(z) & -y_2(z) & 0 \\ -y_1^{'}(z) & -y_2^{'}(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } 2\pi/3 < argz < \pi \\ \\ \begin{pmatrix} -y_2(z) & y_1(z) & 0 \\ -y_2^{'}(z) & y_1^{'}(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } -\pi < argz < -2\pi/3 \\ \\ \begin{pmatrix} y_0(z) & y_1(z) & 0 \\ y_0^{'}(z) & y_1^{'}(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } -2\pi/3 < argz < 0 \end{cases} \tag{5.73}$$

Then

$$\tilde{P}(z) = E_{\vec{n}}(z)\Phi(|\vec{n}|^{2/3}\beta(z)) diag\Big(e^{\frac{1}{2}|\vec{n}|(\lambda_1(z)-\lambda_2(z))}, e^{-\frac{1}{2}|\vec{n}|(\lambda_1(z)-\lambda_2(z))}, 1\Big), \tag{5.74}$$

where $E_{\vec{n}}$ is an analytic prefactor that takes care of the matching condition (5.69).
Explicitly, $E_{\vec{n}}$ is given by

$$E_{\vec{n}} = \sqrt{\pi} M \begin{pmatrix} 1 & -1 & 0 \\ -i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |\vec{n}|^{1/6}\beta^{1/4} & 0 & 0 \\ 0 & |\vec{n}|^{-1/6}\beta^{-1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.75}$$

This approach allows one to do similar constructions works for a parametrix P around the other edge points.

5.2.5 Third transformation of the RH problem: $S \to R$

We put

$$\begin{aligned} R(z) &= S(z)M(z)^{-1} \text{ for } z \text{ outside the disks } D(\pm z_j, r),\ j=1,2 \\ R(z) &= S(z)P(z)^{-1} \text{ for } z \text{ inside the disks} \end{aligned} \tag{5.76}$$

Then R is analytic on $\mathbb{C}\setminus\Gamma_R$, where Γ_R consists of four circles $\partial D(\pm z_j, r), j = 1, 2$, the parts of Γ outside the four disks and the real intervals $(-\infty, -z_1-r), (-z_2+r, z_2-r), (z_1+r, +\infty)$. There are jump relations

$$R_+ = R_- jR \tag{5.77}$$

where

$$\begin{aligned} jR &= MP^{-1} \text{ on the circles, oriented counterclockwise} \\ jR &= MjSM^{-1} \text{ on the remaining parts of } \Gamma_R. \end{aligned} \tag{5.78}$$

From (5.62) it follows that

$$jR = I + O\Big(\frac{1}{|\vec{n}|}\Big) \text{ uniformly on the circle} \tag{5.79}$$

and from (5.37), (5.38) and (5.23) it follows that

$$jR = I + O\Big(e^{-c|\vec{n}|}\Big) \text{ for some } c > 0 \text{ as } |\vec{n}| \to \infty, \text{uniformly on the remaining parts of } \Gamma_R.$$

So we can conclude that

$$jR(z) = I + O\Big(\frac{1}{|\vec{n}|}\Big) \text{ as } |\vec{n}| \to \infty \text{ uniformly on } \Gamma_R \tag{5.80}$$

As $z \to \infty$, we have

$$R(z) = I + O(\frac{1}{z}). \tag{5.81}$$

From (5.77),(5.80),(5.81) and the fact that we can deform the contours in any disired direction, it follows that

$$R(z) = I + O\Big(\frac{1}{|\vec{n}|(|z|+1)}\Big) \text{ as } |\vec{n}| \to \infty \text{ uniformly for } \mathbb{C} \setminus \Gamma_R. \tag{5.82}$$

By Cauchy's theorem

$$R'(z) = O\Big(\frac{1}{|\vec{n}|(|z|+1)}\Big)$$

and thus

$$R^{-1}(y)R(x) = I + R^{-1}(y)(R(x) - R(y)) = I + O\Big(\frac{x-y}{|\vec{n}|}\Big) \tag{5.83}$$

This approch proves simultaneously large $|\vec{n}|$ asyptotics of the (1,1) entry of Y which is equal to the average characteristic polynomial and this polynomial is called a multiple Hermite polynomial for the case of $V(x) = \frac{1}{2}x^2$ on which we are going to discuss in the chapter 6 below.

Chapter 6

Asymptotic behavior of the multiple Hermite polynomials

In this chapter, we use the reference [14].

We have that the $(1,1)$ entry of the solution Y of the RH problem (4.16)-(4.17) is a monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ satisiying (for $r=2$)

$$\int_{-\infty}^{\infty} P_{\vec{n}}(x) x^k \omega_j(x) dx = 0,\ k = 0, 1, \ldots, n_j - 1,\ j = 1, 2.$$

For $\omega_1(x) = e^{-|\vec{n}|(\frac{1}{2}x^2 - ax)}$ and $\omega_2(x) = e^{-|\vec{n}|(\frac{1}{2}x^2 + ax)}$, this polynomial is called a **multiple Hermite polynomial**.
The asymptotic analysis of RH problem yields the strong asymptotics of the multiple Hermite polynomials ($|\vec{n}| \to \infty$ with $|\vec{n}|$ even and $n_1 = n_2$) in every part of the complex plane.
Recall that $P_{\vec{n}}$ is the average characteristic polynomial of the random matrix ensemble

$$\mu_{\vec{n}}(dM) = \frac{1}{Z_{\vec{n}}} e^{-|\vec{n}| Tr(V(M) - AM)} dM.$$

Let us mention that the $(1,1)$ entry to $Y_{1,1}$ satisfies

$$P_{\vec{n}}(z) = Y_{1,1} = \mathbb{E} det(zI - M).$$

Partitioning of the complex plane and derivation of the corresponding multiple Hermite polynomials

We partition the complex plane into 3 regions:

- outside of the lenses and of the disks $D(\pm z_j, r)$ $j = 1, 2$
- inside of the lenses but outside of the disks
- inside of the disks

We derive the large $|\vec{n}|$ asymptotics of the multiple Hermite polynomials in these 3 regions (see fig (5.2)).

6.1 Region outside of the lenses and of the disks

In this region, we obtain the large $|\vec{n}|$ asymptotics of the multiple Hermite polynomials of the form

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \frac{\xi_1^2 - a^2}{\sqrt{(\xi_1^2 - p^2)(\xi_1^2 - q^2)}} e^{-|\vec{n}|\lambda(z)} \Big(1 + O\Big(\frac{1}{|\vec{n}|(|z|+1)}\Big)\Big). \tag{6.1}$$

6.2 Region inside of the lenses but outside of the disks

In upper lens region on $[z_2, z_1]$ we derive the asymptotics of the multiple Hermite polynomials of the form

$$\begin{aligned} P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} &= \Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_1(z) + |\vec{n}|l_1} \\ &+ \Big(\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_2(z) + |\vec{n}|l_1} \end{aligned} \tag{6.2}$$

where

$$\lambda_k(z) = \int_{z_1}^{z} \xi_k(s)ds$$

Similarly in the lower region on $[z_2, z_1]$ we derive the multiple Hermite polynomials of the form

$$\begin{aligned} P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} &= \Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_1(z) + |\vec{n}|l_1} \\ &- \Big(\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O(\frac{1}{|\vec{n}|})\Big) e^{-|\vec{n}|\lambda_2(z) + |\vec{n}|l_1}. \end{aligned} \tag{6.3}$$

For $z = x$ real, $x \in [z_2 + r, z_1 - r]$, both (6.2) and (6.3) can be written in the form

$$P_{\vec{n}}(x)e^{-\frac{|\vec{n}|}{2}x^2} = A(x)cos[|\vec{n}|Im\lambda_{1+}(x) - \varphi(x)] + O(\frac{1}{|\vec{n}|})e^{-|\vec{n}|Re\lambda_{1+}(x)+|\vec{n}|l_1} \tag{6.4}$$

where

$$A(x) = 2\Big|\frac{\xi_{1+}^2(z) - a^2}{\sqrt{(\xi_{1+}^2(z) - p^2)(\xi_{1+}^2(z) - q^2)}}\Big| \tag{6.5}$$

and

$$\varphi(x) = arg\frac{\xi_{1+}^2(z) - a^2}{\sqrt{(\xi_{1+}^2(z) - p^2)(\xi_{1+}^2(z) - q^2)}} \tag{6.6}$$

By equation (5.16) we have

$$\int_{z_1}^{x} Im\xi_{1+}(s)ds \equiv Im\lambda_{1+}(x) = \pi\int_{z_1}^{x}\rho(s)ds. \tag{6.7}$$

By using this equation (6.7) we can rewrite (6.3) in terms of the eigenvalue density function $\rho(x)$ as follows

$$P_{\vec{n}}(x)e^{-\frac{|\vec{n}|}{2}x^2} = A(x)cos[|\vec{n}|\pi\int_{z_1}^{x}\rho(s)ds - \varphi(x)] + O(\frac{1}{|\vec{n}|})e^{-|\vec{n}|Re\lambda_{1+}(x)+|\vec{n}|l_1}. \tag{6.8}$$

Clearly, equation (6.8) displays the oscillating behavior of $P_{\vec{n}}$ on the interval $[z_2+r, z_1-r]$. It also shows that the zeros of $P_{\vec{n}}$ are asymptotically distributed like $\rho(x)dx$, he limiting probability distribution of eigenvalues.
Similar formula can be derived on the interval $[-z_1 + r, -z_2 - r]$ as well.

6.3 Region inside of the disks

Consider the disk $D(z_1, r)$. In this region, we derive multiple Hermite polynomials in the four regions.

6.3.1 In region I and IV

In this region, we obtain the asymptotics of the multiple Hermite polynomials of the form

$$\begin{aligned} P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[&|\vec{n}|^{1/6}B(z)Ai(|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) + |\vec{n}|^{-1/6}C(z) \\ &Ai'(|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1} \end{aligned} \tag{6.9}$$

where

$$B(z) = \beta(z)^{1/4}\Big(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}}\Big) \tag{6.10}$$

and

$$C(z) = \beta(z)^{-1/4}\Big(- \frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}}\Big) \tag{6.11}$$

The following results in region II and III are an other contribution in this work.

6.3.2 In region II

In this region, we find the asymptotics of the multiple Hermite polynomials of the form

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[- \omega|\vec{n}|^{1/6}B(z)Ai(\omega|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) - \omega^2|\vec{n}|^{-1/6}C(z) \\ Ai^{'}(\omega|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1} \tag{6.12}$$

where $B(z)$ and $C(z)$ are defined by (6.10) and (6.11) respectively.

6.3.3 In region III

Similarly in this region, we find the asymptotics of the multiple Hermite polynomials of the form

$$P_{\vec{n}}(z)e^{-\frac{|\vec{n}|}{2}z^2} = \sqrt{\pi}\Big[- \omega|\vec{n}|^{1/6}B(z)Ai(\omega^2|\vec{n}|^{2/3}\beta(z))\Big(I + 0(\tfrac{1}{|\vec{n}|})\Big) - \omega^4|\vec{n}|^{-1/6}C(z) \\ Ai^{'}(\omega^2|\vec{n}|^{2/3}\beta(z))\Big(I + O(\tfrac{1}{|\vec{n}|})\Big)\Big]e^{-|\vec{n}|\alpha(z)+|\vec{n}|l_1} \tag{6.13}$$

where $B(z)$ and $C(z)$ are again defined by (6.10) and (6.11) respectively.
It can be verified that the functions $B(z)$ and $C(z)$ are analytic in $D(z_1, r)$.
This approach allows one to derive the forms of the multiple Hermite polyonomials in all other disks $D(\pm z_j, r)$ $j = 1, 2$ as well.

Conclusion and outlooks

To conclude, let us revisit the main results we can across in this thesis. Recall that the main focus of our study was to investigate the multiple Hermite polynomials by their properties, to show the relationship with the random matrices with external source and to analyze those polynomials by using the steepest descent method of Deif and Zhou based on the RH problem as $z \to \infty$.
We first recall the classical orthogonal polynomials, introduce the multiple orthogonal polynomials in general and prove some properties of the multiple Hermite polynomials which allowed us to correct the Theorem 2.4 of D.W. Lee in Theorem 3.0.3 concerning the recurrence relation by the generating function of the multiple Hermite polynomials .
We show the relationship between the multiple Hermite polynomials with the random matrices with external source for the Gaussian case.
We analyze the RH problem by the steepest descent method of Deif and Zhou based on the RH problem by using some transformations.
Lastly, we relate those transformations to characterize the asymptotic behavior of the multiple Hermite polynomials.

While we hope this analysis is complete, it would be unfair to label it fully conclusive. There are some open questions that the author feels are left unanswered and therefore deserve further researching. In particular, we would like to mention the two of them:

- It would be a great step to find the asymptotic behavior of the multiple Hermite polynomials in all other disks such as $D(z_2, r)$, $D(-z_1, r)$ and $D(-z_2, r))$ of the complex plane

- It would also be important to analyze the asymptotic behavior of the multiple Laguerre and the multiple Jacobi polynomials by using the steepest descent method of Deif and Zhou based on the RH problem.

Bibliography

[1] A.I Aptekarev, A.Branquinho & W.Van Assche: *Multiple orthogonal polynomial for classical weigths*, Trans.Amer.Math.Soc.355 (2004),3887-3914.

[2] D.W.Lee: *Properties of multiple Hermite and multiple Laguerre polynomials by generating function*, Integral Transforms Spec.Funct.18(2007)no11-12,855-869.

[3] E.Brézin & S.Hikami: *Level spacing of random matrices in an external source*, Phys.Rev.E 58(1998),7176-7185.

[4] E.Brézin & S.Hikami: *Spectral form factor in random matrix theory*, Phys.Rev.E 55(1997),4067-4083.

[5] E.Daems & A.B.J Kuijlaars: *Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions* , J.Approx.Theory 146(2007)no1,91-114.

[6] G.Filipuk, M.Haneczok & W.Van Assche: *Computing recurrence coefficient of multiple orthogonal polynomials* Numer.Algorithms 70(2015), no3,519-543.

[7] Harish-Shandra:*Differential operators on a semisimple Lie algebra*, Amer.J.Math.79(1957),87-120.

[8] K.Mahler: *Perfect systems*, Compositio.Math.19(1968),95-166.

[9] L.A Pastur: *The spectrum of random matrices (Russian)*, Teoret.Mat.Fiz.10(1972),102-112.

[10] M.E.H Ismail: *Classical and quantum orthogonal polynomials in one variable, Encyclopedia of Mathematics and its applications 98*, Cambridge University Press(2005).

[11] P.Deift: *Orthogonal matrices and random matrices: A Riemann-Hilbert approach, Courant Lecture Notes in Mathematics,vol.3*, Amer.Math.Soc.,Rhode Island(1999).

[12] P.Deift & X.Zhou: *Steepest descend method for oscillatory Riamann-Hilbert problems, Asymptotics for the mKdV equation*, Ann.Math.137(1993),295-368.

[13] P.M.Bleher & A.B.J.Kuijlaars: *Integral representations for multiple Hermite and multiple Laguerre polynomials*, Ann.Inst.Fourier,Grenoble 55(2005),no6,2001-2014.

[14] P.M.Bleher & A.B.J.Kuijlaars: *Large n limit on Gaussian random matrices with external source,Part I*, Commun.Math.Phys.252(2004),43-76.

[15] P.M.Bleher & A.B.J.Kuijlaars: *Random matrices with external source and multiple orthogonal polynomials*, Internat.Math.Research Notices 2004(2004),109-129.

[16] P.Zinn-Justin: *Random Hermitian matrices in an external field*, Nuclear Phys.B496(1997),725-732.

[17] S.Hikami & E.Brézin: *Spectral form factor in a random matrix theory*, Phys.Rev.E55(1997),4067-4083.

[18] W.Van Assche: *Physique Théorique et Mathematique*, Notes du cours-Université du Burundi(2018).

[19] W.Van Assche: *Nearest neighbor recurrence relations for multiple orthogonal polynomials*, J.Approx.Theory 163(2011),no10,1427-1448.

[20] W.Van Assche & E.Coussement: *Some classical multiple orthogonal polynomials*, J.Comput.Appl.Math.127(2011)317-347. 163(2011),no10,1427-1448.